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Survey paper

SOME APPLICATIONS OF POTENTIALS AND APPROXIMATIVE INVERSE OPERATORS IN MULTI-DIMENSIONAL FRACTIONAL CALCULUS

I.   Introduction

There are known several approaches in fractional calculus to the definition of fractional integration and differentiation of functions on R^n . One of these approaches deals with complex powers $[P(\mathcal{D})]^\alpha$, where $P(\mathcal{D})$ is a partial differential operator in R^n with constant coefficients (see,for example, the books [37] and [38]). In this survey we treat fractional integrals and derivatives in just the same way. We consider complex powers of second order differential operators of the form

$$-\Delta_{x'} + c \cdot \mathcal{D}, \quad \mathcal{D} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right), \tag{1.1}$$

where

$$c \in \mathbf{C}^n, \quad x' = (x_1, \dots, x_k), \quad k \le n - 1 \ .$$

The class of the operators (1.1) contains in particular examples of such operators as hypoelliptic operators and generalized Schrödinger operators, which are known to be of essentially different nature. Negative ($\Re\alpha < 0$) powers of these operators are realized as potentials, positive powers - inverse to negative- as approximative inverse operators (AIO), within the framework of the spaces $L_p(R^n)$, which can be represented as hypersingular integrals (HSI) in case of nice functions.

We also consider some other problems of potential theory, such as $L_p \to L_q-$estimates for potentials with oscillating kernels or symbols. It should be noted that we deal with potentials, having "bad" properties of their kernels and (or) symbols. Their kernels have singularities, "spread" over hyperplanes or spheres (which may be even locally non-integrable). Their symbols degenerate on different sets in R^n (in the most general case - on a set of null measure in R^n) or have singularities, "spread" over spheres, paraboloids and other sets. Such "bad" properties of kernels and symbols produce natural difficulties in the investigation of these potentials.

We observe that the previous surveys [35] and [21] covered, in particular, investigations of complex powers of some special differential operators (in particular, the classical operators of mathematical physics). In these surveys some applications of HSI and AIO to the inversion of potential-type operators were also given.

The main results covered in this survey were obtained during the last three years in the research group, guided by the second author at the Rostov State University.

II. Complex powers of second order differential operators with complex coefficients in lower

In the papers [1],[2] (see also [21]) complex powers of the operators

$$-\Delta + c \cdot \mathcal{D}, \quad c \in \mathbf{C}^n \tag{2.1}$$

within the framework of the space $L_p \equiv L_p(R^n)$ were investigated. In this Section we deal with complex powers of the operator (1.1), which is more difficult in comparison with the operator (2.1), especially when $c = ib$, $b \in R^n$.

Negative powers I_c^α of the operator (1.1) are defined on nice function $\varphi(x)$ via Fourier transforms as follows:

$$(\mathcal{F} I_c^\alpha \varphi)(\xi) \ := \ \left(|\xi'|^2 - ic \cdot \xi\right)^{\frac{-\alpha}{2}} (\mathcal{F}\varphi)(\xi), \quad \Re\alpha > 0 . \tag{2.2}$$

We realize the operator I_c^α as a potential with explicitly written kernels and treat negative powers

$$(-\Delta_{x'} + c \cdot \mathcal{D})^{-\frac{\alpha}{2}}$$

as these potential-type convolution operators within the framework of L_p-spaces.

We start with the case $c \in R^n$.

2.1. Complex powers of hypoelliptic operators in L_p-spaces

We assume that $c \in R^n$, $c_n \neq 0$ (the case $c_n = 0$ reduces to the case of elliptic operators (2.1) in R^{n-1}). Let $\Re\alpha > 0$,

$$k_c^\alpha(x) = \frac{2^{1-n}\pi^{\frac{1-n}{2}}}{\Gamma(\frac{\alpha}{2})|c_n|} \left(\frac{x_n}{c_n}\right)^{\frac{\alpha-n-1}{2}} exp \left(\frac{x' \cdot c'}{2} - \frac{|x'|^2}{4}\frac{c_n}{x_n} - \frac{|c'|^2}{4}\frac{x_n}{c_n}\right) \tag{2.3}$$

if $\frac{x_n}{c_n} > 0$ and

$$k_c^\alpha(x) = 0$$

if $\frac{x_n}{c_n} \leq 0$. Then

$$(I_c^\alpha \varphi)(x) = \int_{R^n} k_c^\alpha(t)\varphi(x - t)dt \tag{2.4}$$

for nice function $\varphi(x)$. The convolution operator (2.4) is defined on functions $\varphi(x)$ in L_p for $0 < \Re\alpha < n + 1$, $1 \leq p < \frac{n+1}{\Re\alpha}$ and is bounded from L_p into L_q, $q = \frac{(n+1)p}{n+1-p\Re\alpha}$ if $1 < p < \frac{n+1}{\Re\alpha}$ (see [3] and [5]).

Within the framework of the method of AIO, positive powers of the operator (1.1) were realized in [3] and [5] as follows. Let

$$\mathbb{D}_{c,\varepsilon}^\alpha f = h_{c,\varepsilon}^\alpha * f , \tag{2.5}$$

where

$$h_{c,\varepsilon}^\alpha(x) = \mathcal{F}^{-1}\left[\left(|\xi'|^2 - ic \cdot \xi\right)^{\frac{\alpha}{2}} \left(\frac{c \cdot \xi}{c \cdot \xi + i\varepsilon}\right)^m exp\left(-\varepsilon|\xi|^2\right)\right](x) \tag{2.6}$$

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$m = 0$ if $1 < \Re\alpha < n + 1$ or $0 < \Re\alpha \le 1$, $\quad 1 < p < \frac{2(n+1)}{2\Re\alpha + n + 1}$ and $m = 1$ if $0 < \Re\alpha \le 1$, $\frac{2(n+1)}{2\Re\alpha + n + 1} < p < \frac{(n+1)}{\Re\alpha}$.

Theorem 2.1. ([3],[5]) *Let* $0 < \Re\alpha < n + 1$, $\quad f = I_c^\alpha \varphi$, $\varphi \in L_p$, $1 \le p < \frac{(n+1)}{\Re\alpha}$. *Then*

$$\mathbb{D}_c^\alpha I_c^\alpha \varphi = \varphi \ ,$$

where

$$\mathbb{D}_c^\alpha f = \overset{(L_p)}{\underset{\varepsilon \to 0}{\lim}} \ \mathbb{D}_{c,\varepsilon}^\alpha f \tag{2.7}$$

The relation (2.7) is also valid with the almost everywhere limit. Moreover,

$$I_c^\alpha(L_p) = \left\{ f : f \in L_q, \ \mathbb{D}_c^\alpha f \in L_p, \ 1 < p < \frac{(n+1)}{\Re\alpha}, \ q = \frac{(n+1)p}{n+1 - p\Re\alpha} \right\}$$

This theorem gives an explicit expression for positive powers of the operator (2.1), within the framework of L_p-spaces, as well as a description of domains of these powers. It was also generalized in [3] and [5] to the case of an arbitrary second order hypoelliptic operator in R^n with real coefficients.

2.2. Complex powers of some strongly degenerating second order differential operators with real coefficients

Potentials realizing negative powers of the operator (1.1) may be reduced, in the case $k < n - 1$, $c \in R^n$, to the known ones (in R^{k+1} or R^k). In the case $c'' \neq 0$, $c'' = (c_{k+1}, \ldots, c_n)$, the operator I_c^α may be written in the form

$$(I_c^\alpha \varphi)(x) = U^{-1} I_{(U^{-1}c)'_*, \ x'_*}^{\alpha,k} U\varphi(x), \tag{2.8}$$

where $x'_* = (x_1, \ldots, x_{k+1})$, $I_{a'_*, x'_*}^{\alpha,k}$ is the operator (2.4) in R^{k+1}, applied in the variable x'_*, U is a rotation in R^n such that $Uc = (c_1, \ldots, c_k, |c''|, 0, \ldots, 0)$

If $c'' = 0$ then

$$(I_c^\alpha \varphi)(x) = J_{c'}^{\alpha,k} \varphi(\cdot, x'')(x'), \tag{2.9}$$

where $J_{c'}^{\alpha,k}$ is a partial potentials in R^k, realizing negative powers of the operator (2.1) and applied in the variable x', see [1] and [2] for the explicit form of these potentials.

Using the results of the papers [1] and [2] and Theorem 2.1, one can construct the inversion of potential I_c^α (on the basis of (2.8), (2.9)) and describe its range (see [4] for details).

2.3. The case $c \in \mathbf{C}^n$, $k = n - 1$

Passing to the case of complex coefficients in lower terms, we consider here only the most interesting situations (see [6] for other cases).

Let us consider the case $\Re c_n \neq 0$ at first. The kernel $k_c^\alpha(x)$ of the potential I_c^α, realizing negative powers of the operator (1.1), has the form (2.3) with $\sum_{j=1}^{n-1} c_j^2$ instead of $|c'|^2$ and $\sqrt{c_n^2}$ instead of $|c_n|$ if $\frac{x_n}{\Re c_n} > 0$ and equals zero if $\frac{x_n}{\Re c_n} \le 0$. The condition

$$\Im c = d \, \Re c \quad \text{for some} \quad d \in R^1 \backslash \{0\} \tag{2.10}$$

appears here. If this condition fails, then the kernel $k_c^\alpha(x)$ becomes non-integrable at infinity with respect to any power $q \geq 1$; in fact, it exponentially grows at infinity. In accordance with (2.10) we consider the case $c = \lambda a$, $\lambda = 1 + id$, $d \in R^1\backslash\{0\}$, $a \in R^n$, $a_n \neq 0$.

Let $\mathbb{D}_{c,\varepsilon}^\alpha$ be the operator (2.5) with the kernel (2.6) in which the function $\left(\frac{c\cdot\xi}{c\cdot\xi+i\varepsilon}\right)^m$ is substituted by $\left(\frac{a\cdot\xi}{a\cdot\xi+i\varepsilon}\right)^m$ with $m = 0$ if $\frac{\Re\alpha}{2} > n - 1$ and $m > n - 1 - \frac{\Re\alpha}{2}$ if $\frac{\Re\alpha}{2} \leq n - 1$. An explicit expression for positive powers of the operator (1.1) (within the framework of L_p-spaces) may be obtained and the domains of these powers may be also described in terms of the operator (2.7). The corresponding results are formulated in the same way as in the case of hypoelliptic operators (see Theorem 2.1).

The most difficult is the case $\Re c_n = 0$ with $\Im c_n \neq 0$, which is essentially different from the described above. Let I_c^α be the operator (2.4) with the kernel

$$k_c^\alpha(x) = \begin{cases} \frac{2^{1-n}\pi^{\frac{1-n}{2}}exp\left(\frac{(n-\alpha-1)\pi i}{4}\right)}{|\Im c_n|}\left(\frac{x_n}{\Im c_n}\right)^{\frac{\alpha-n-1}{2}} exp\left(\frac{x'\cdot c'}{2} - i\frac{|x'|^2}{4}\frac{\Im c_n}{x_n} - \frac{\sum_{j=1}^{n-1}c_j^2}{4}\frac{x_n}{\Im c_n}\right) \\ \qquad\qquad if \quad \left(\frac{x_n}{\Im c_n}\right)^{\frac{\alpha-n-1}{2}} > 0 \end{cases}$$

$$\text{(2.11)}$$

$$and \quad k_c^\alpha(x) = 0 \quad if \quad \left(\frac{x_n}{\Im c_n}\right)^{\frac{\alpha-n-1}{2}} \leq 0 .$$

It is seen from (2.11) that $k_c^\alpha(x)$ exponentially grows at infinity if $\Re c' \neq 0$. Therefore we assume below that $c = ib$, $b \in R^n$, $b_n \neq 0$.

For $\omega(x)$ in a special class of test functions (invariant for the operator I_{ib}^α) it was shown in [6] that

$$\mathcal{F}(I_{ib}^\alpha\omega)(\xi) = \frac{(\mathcal{F}\omega)(\xi)}{(|\xi'|^2 + b\cdot\xi + i0)^{\frac{\alpha}{2}}}, \quad \Re\alpha > 0 . \qquad (2.12)$$

The potential $I_{ib}^\alpha\omega$ is interpreted in the "usual" sense - as the integral (2.4) over R^n - if $\Re\alpha > n - 1$ and in the sense of analytic continuation of the integral (2.4) if $0 < \Re\alpha \leq n - 1$ (see [6]). By virtue of (2.12) we conclude that the operators I_{ib}^α really realize negative powers of the operator (1.1).

Inversion of the potential $I_{ib}^\alpha\omega$, $\Re\alpha > 0$, was constructed in [6] on nice functions $\omega(x)$ in the form of the following HSI

$$(\mathcal{D}_{ib}^\alpha f)(x) = \frac{1}{d_{n,l}(\alpha)}\int_{R^n}(\Delta_y^l f)(x)k_{ib}^{-\alpha}(y)dy,$$

where

$$(\Delta_y^l f)(x) = \sum_{j=0}^l c_j f(x - a^j y), \quad a > 1$$

is the generalized difference of the function $f(x)$ (see [6] for details).

The integral $I_{ib}^\alpha\varphi$ does not converges, generally speaking, for $\varphi \in L_p$, so that the potential $f = I_{ib}^\alpha\varphi$ is treated in this case in the distributional sense, see [6]. A description

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of the range $I_{ib}^\alpha(L_p)$ seems to be a problem, but in [6] the authors succeeded to describe the spaces $I_{ib}^\alpha(L_p) \bigcap L_r = I_{p,r}^\alpha$. We note that these spaces are some analogs of the spaces $L_{p,r}^\alpha(R^n)$ of Riesz potentials, introduced and investigated in [28] and [29], see also [34],[37] and [38].

To formulate one of the main results, presented in this survey, we denote

$$
\mathbb{D}_{ib}^\alpha f = \begin{cases} \overset{(L_p)}{\underset{\varepsilon\to 0}{\lim}} \overset{(L_2)}{\underset{\delta\to 0}{\lim}} \left(h_{ib,\varepsilon,\delta}^{\alpha,n-1} * f\right)(x) & if \quad 0 < \Re\alpha < 2n, \quad \alpha \neq 2,4,\ldots,2(n-1) \\[4mm] \overset{(L_p)}{\underset{\varepsilon\to 0}{\lim}} \left(h_{ib,\varepsilon,0}^{\alpha,0} * f\right)(x) & if \quad \Re\alpha \geq 2n, \quad or \quad \alpha = 2,4,\ldots,2(n-1), \end{cases} \tag{2.13}
$$

where

$$
h_{ib,\varepsilon,\delta}^{\alpha,\theta}(x) = \mathcal{F}^{-1}\left[\frac{(|\xi'|^2 + b\cdot\xi + i0)^{\frac{\alpha}{2}} + \theta}{(|\xi'|^2 + b\cdot\xi + i\delta)^\theta} exp\left(-\varepsilon|\xi|^2\right)\right](x) ,
$$

$$
\theta = \begin{cases} n-1 & if \quad 0 < \Re\alpha < 2n, \quad \alpha \neq 2,4,\ldots,2(n-1) \\[3mm] 0 & if \quad \Re\alpha \geq 2n, \quad or \quad \alpha = 2,4,\ldots,2(n-1). \end{cases}
$$

Theorem 2.2. ([6]) *Let $\Re\alpha > 0$, $1 \leq p,r \leq 2$. Then*
1) for $f \in I_{p,r}^\alpha$ with $f = I_{ib}^\alpha\varphi$, $\varphi \in L_p$, we have

$$
\mathbb{D}_{ib}^\alpha f = \varphi,
$$

where $\mathbb{D}_{ib}^\alpha$ is the operator (2.13); the limit in L_p-norm in (2.13) may be also replaced by the almost everywhere limit;

2) $I_{p,r}^\alpha = \{f : f \in L_r, \quad \mathbb{D}_{ib}^\alpha f \in L_p\}$, where $\mathbb{D}_{ib}^\alpha$ is the operator (2.13).

We observe that the case $c = (0,\ldots,0,i)$ (the case of Shrödinger operator) was considered in [22]. There is also the paper [23] in which the spaces of $L_{p,r}^\alpha$−type , connected with the wave and the Klein-Gordon-Fock operators were investigated (see also the survey [21]).

III. $L_p - L_q-$ estimates for potentials with oscillating kernels or symbols

In this section we consider some potentials with oscillating kernels or symbols. Our goal is to construct a convex set on the $(\frac{1}{p},\frac{1}{q})$-plane such, that the corresponding potential is bounded from L_p into L_q if the point $(\frac{1}{p},\frac{1}{q})$ belongs to this set. In some cases we succeed to construct the $\mathcal{L}-$ characteristic of the potential under the consideration, that is, we give an explicit description of all pairs $(\frac{1}{p},\frac{1}{q})$ for which the operator is bounded from L_p into L_q. We note that construction of the $\mathcal{L}$-characteristic of the potential is an interesting and difficult problem itself. Also, knowledge of the $\mathcal{L}$-characteristic is very useful for description of ranges of potentials in non-elliptic case, see Subsection 4.4.

3.1. $L_p \to L_q$-estimates for Riesz-type potentials with oscillating characteristics

To formulate the corresponding result for the operator

$$(K^\alpha f)(x) = \int_{R^n} \frac{e^{i|y|}}{|y|^{n-\alpha}} f(x-y) dy, \quad 0 < \alpha < n, \quad n \geq 2, \qquad (3.1)$$

we introduce the following notation:

$\mathcal{L}(A) = \left\{ (\frac{1}{p}, \frac{1}{q}) \in [0,1] \times [0,1] : \|A\|_{L_p \to L_q} < \infty \right\};$

$(A, B, C, \ldots, K)$ is an open polygon;

$[A, B, C, \ldots, K]$ is its closure;

(A, B) is an open interval;

$a = 1 - \frac{\alpha}{n}, \quad b = \frac{2n + \alpha(n-1)}{n(n+1)}, \quad d = \frac{1 + 2\alpha - \frac{n-1}{n+1}}{2}, \quad e = 1 - \frac{(1-\frac{\alpha}{n})(n-1)}{n+3}, \quad k = \frac{n+3}{2(n+1)};$

$A' = (1-a, 0), \quad B' = (1-a, 1-b), \quad C' = (k, k), \quad G' = (1-a, 1-e), \quad H' = (1-a, 1-a);$

$A = (1, a), \quad B = (b, a), \quad C = (1-k, 1-k), \quad D = (d, 1-d), \quad E = (1, 0), \quad G = (e, a), \quad H = (a, a)$

Theorem 3.1. ([24]) *I. The following imbeddings are valid:*

a) $(A', B', B, A) \bigcup (A', A') \bigcup \{D\} \subset \mathcal{L}(K^\alpha) \quad if \quad n = 2, \quad \frac{1}{2} \leq \alpha < 2;$

b) $(A', H', H, A) \bigcup (A', A) \bigcup (H', H) \subset \mathcal{L}(K^\alpha) \quad if \quad n = 2, \quad 0 < \alpha < \frac{1}{2}$

$\quad or \quad n = 3, \quad 0 < \alpha < \frac{3}{4} \quad or \quad n > 3, \quad 0 < \alpha < \frac{n-1}{4};$

c) $(A', G', G, A) \bigcup (A', A) \subset \mathcal{L}(K^\alpha) \quad if \quad n \geq 3, \quad \frac{n(n-1)}{2(n+1)} \leq \alpha < \frac{n-1}{2};$

d) $(A', G', D, G, A) \bigcup (A', A) \subset \mathcal{L}(K^\alpha) \quad if \quad n \geq 3, \quad \frac{n-1}{2} \leq \alpha < n;$

e) $[A', C', C, A] \backslash (\{A'\} \bigcup \{A\}) \subset \mathcal{L}(K^\alpha) \quad if \quad n > 3, \quad \frac{n-1}{4} \leq \alpha < \frac{n}{n-1}.$

II. The set $\mathcal{L}(K^\alpha)$ does not contain:

a) the points, lying on the segment $[H, A]$ and above it;

b) the points, lying on the segment $[A', H']$ and to the left of it;

c) the points, lying above the straight line BB', when $\frac{n-1}{2} < \alpha < n;$

d) the points, belonging to the set $[A', A, E] \backslash (A', A).$

We observe that the study of $L_p \to L_q$-boundedness of potentials with oscillating kernels are at the very beginning. Besides the operator K^α, there were considered only two classes of "specific" oscillation, namely, generated by the Bessel function which give the Bochner-Riesz means, see [8], and by the Hankel function producing the acoustic potentials, see [20].

3.2. On the $\mathcal{L}$-characteristic of some potentials with oscillating symbol

in this Subsection we construct the $\mathcal{L}$-characteristic of the operator A^α with the symbol

$|\xi|^{-\alpha} exp\,(i|\xi|)$. This operator was realized in [43] on functions $\varphi(x) \in L_p$ in the form

$$(A^\alpha \varphi)(x) = \int_{R^n} \Omega_\alpha(|y|)\varphi(x-y)dy, \tag{3.2}$$

$\frac{n-1}{2} < \alpha < n$. An explicit expression for the kernel $\Omega_\alpha(|y|)$ in terms of the Gauss hypergeometric function is given in [43]. We only note that $\Omega_\alpha(|y|)$ is continuous in $R^n \backslash S^{n-1}$, where S^{n-1} is the unit sphere in R^n, and

$$\Omega_\alpha(|y|) \sim c\,(1-|y|)^{\alpha - \frac{n+1}{2}}, \quad \frac{n-1}{2} < \alpha < \frac{n+1}{2};$$

$$\Omega_\alpha(|y|) \sim c\ln|1-|y||, \quad \alpha = \frac{n+1}{2} \quad for \quad |y| \to 1;$$

and

$$\Omega_\alpha(|y|) \sim c|y|^{\alpha-n}, \quad for \quad |y| \to \infty\,.$$

Theorem 3.2. ([10]) *Let $\frac{n-1}{2} < \alpha < n$. The operator A^α is bounded:*
a) from L_p into L_q, $1 < p \le q < \infty$ if and only if $\frac{1}{q} \le \frac{1}{p} - \frac{\alpha}{n}$ and

$$\frac{1}{p} + \frac{1}{q} \le 1, \ \frac{1}{p} - \frac{n}{q} \le \alpha - \frac{n-1}{2} \quad or \quad \frac{1}{p} + \frac{1}{q} \ge 1, \ \frac{n}{p} - \frac{1}{q} \le \alpha + \frac{n-1}{2};$$

b) from L_1 into L_q, $1 \le q \le \infty$ if and only if $\frac{1}{q} < 1 - \frac{\alpha}{n}$ and $-\frac{1}{q} \le \alpha + \frac{n-1}{2}$;
c) from L_p into L_∞, $1 < p \le \infty$ if and only if $\frac{\alpha}{n} < \frac{1}{p} < \alpha - \frac{n-1}{2}$.

There is a number of papers devoted to $L_p \to L$-estimates for operators with oscillating symbols, see [12],[13],[26],[27],[40] and [42], but all the symbols in these papers do not have singularities, being smooth functions.

As an application of Theorem 3.2 we consider the convolution operators A_ω^α with symbol of the form

$$\omega(\xi')|\xi|^{-\alpha} exp\,(i\gamma|\xi|), \quad \xi' = \frac{\xi}{|\xi|}, \quad \gamma > 0$$

Theorem 3.3. ([10]) *Let $\frac{n-1}{2} < \alpha < n$, $\omega \in C^m(S^{n-1})$, $m > \frac{3n}{2}$. and satisfies the ellipticity condition: $\omega(\sigma) \ne 0$, $\sigma \in S^{n-1}$. Then*

$$\mathcal{L}(A_\omega^\alpha) = \mathcal{L}(A^\alpha)$$

3.3. The case of potentials with the Hankel function in their symbols

We consider the operator

$$(H^\alpha \varphi)(x) = c_{n,\alpha} \int_{R^n} \left(1 - |t|^2 + i0\right)^{\frac{\alpha}{2}-1} \varphi(x-t)dt, \quad 0 < \alpha < 2, \tag{3.3}$$

$$c_{n,\alpha} = \frac{\Gamma(1-\frac{\alpha}{2})exp\,(\frac{-i\pi(\alpha-2)}{2})}{i\pi^{\frac{n}{2}+1}2^{\frac{\alpha+n}{2}-1}},$$

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with the symbol

$$h_\alpha(|\xi|) = |\xi|^{-\frac{n+\alpha-2}{2}} H_{\frac{n+\alpha-2}{2}}(|\xi|) \, , \tag{3.4}$$

where $H_\nu^{(1)}(z)$ is the first Hankel function. Because of the the function $H_{\frac{n+\alpha-2}{2}}^{(1)}(|\xi|)$, the symbol $h_\alpha(|\xi|)$ has oscillation.

Theorem 3.4. ([19]) *Let A, B and C be the points on $\left(\frac{1}{p}, \frac{1}{q}\right)$-plane defined as*

$$A = \left(\frac{4 - \alpha - 2n}{2(1 - n)}, \ \frac{(\alpha - 2)(n - 2)}{2n(1 - n)} \right),$$

$$B = \left(\frac{(n - 2)(\alpha + 2n) + 4}{2n(n - 1)}, \ \frac{(2 - \alpha)}{2(1 - n)} \right),$$

$$C = \left(\frac{(\alpha + 2n}{2(n + 1)}, \ \frac{(2 - \alpha)}{2(n + 1)} \right).$$

Then

$$\mathcal{L}(H_\omega^\alpha) = [A, B, C] \quad if \quad n \geq 3$$

and

$$\mathcal{L}(H_\omega^\alpha) = [A, B, C] \backslash (\{A\} \bigcup \{B\}) \quad if \quad n = 2 \, .$$

The statement of Theorem 3.4 is also valid in the case $2 - n < \alpha < 0$, $\alpha \neq -2, -4, \ldots$, if the operator H^α is interpreted as analytic continuation of the integral in the right-hand side of (3.3). Such analytic continuation into the domain $2 - n < \Re\alpha < 0$, $\alpha \neq 0, -2, -4, \ldots$, may be constructed by integrating by parts in the form

$$(H^\alpha \varphi)(x) = \int_{R^n} \left(1 - |\rho|^2 + i0\right)^{\frac{\alpha}{2} + k - 1} \left[\frac{\partial}{\partial \rho} \frac{1}{\rho} \right]^k \left(\rho^{n-1} S_\varphi(x, \rho) \right) d\rho \, , \quad \varphi \in \mathcal{S} \tag{3.5}$$

($\mathcal{S}$ is the Schwarz class of rapidly decreasing smooth functions),

$$k = \left[\frac{n-1}{2} \right], \quad c_{n,\alpha}^k = \frac{c_{n,\alpha}}{\alpha(\alpha + 2) \ldots (\alpha + 2(k-1))}, \quad S_\varphi(x, \rho) = \int_{S^{n-1}} \varphi(x - \rho\sigma) d\sigma.$$

Theorem 3.5. ([19]) *Let $2 - n < \alpha < 0$, $\alpha \neq -2, -4, \ldots$ Then the operator (3.5) may be extended to a bounded operator from L_p into L_q if and only if $(\frac{1}{p}, \frac{1}{q}) \in [A, B, C]$.*

IV. Inversion of some potentials with singularities of their kernels on a sphere

An actual problem of potential theory is the inversion problem for potentials inversion of some potentials with singularities of their kernels "spread" over different sets in R^n. The investigations in this direction are at the very beginning. In the previous survey

[21] the authors dealt with the application of AIO's method to special-type potentials only. In this Section, within the framework of AIO's method, we mention results on the inversion of wide classes of potentials with singularities of their kernels on a sphere $|t| = \gamma$ is constructed both in elliptic and non-elliptic cases. These potentials are defined as convolution operators with symbols of the form

$$m^\alpha_{\theta,\gamma}(\xi) = \frac{\theta(\xi)}{|\xi|^\alpha} exp\ (i\gamma|\xi|)\ ,\quad \gamma > 0\ , \tag{4.1}$$

where θ is a smooth function.

We also construct the inversion of well-known operators

$$(M^\alpha\varphi)(x) = \frac{\Gamma(\frac{n}{2} + \alpha)}{\pi^{\frac{n}{2}}\Gamma(\alpha)} \int_{|y|<1} \left(1 - |y|^2\right)^{\alpha-1} \varphi(x - y)dy\ ,\quad \alpha > 0 \tag{4.2}$$

with the symbols

$$m^\alpha(|\xi|) = \Gamma\left(\frac{n}{2} + \alpha\right)\left(\frac{\xi}{2}\right)^{1-\frac{n}{2}-\alpha} J_{\frac{n}{2}+\alpha-1}(|\xi|)\ , \tag{4.3}$$

where $J_\nu(z)$ is the Bessel function and their modifications. We also give the description of the ranges of these operators. We observe that the mentioned results, related to the description of the ranges of potentials with "spread" singularities of their kernels in non-elliptic case, are the first ones in this direction - as is seen from (3.3), the symbol $m^\alpha(|\xi|)$ degenerates on an infinite union of spheres. In this connection we can refer only to the papers [15] and [21], in which a description of the ranges of fractional "telegraph" potentials, that is, of negative powers of "telegraph" operator, was given, which corresponds to the case when the kernel of these potentials have singularities on a cone), but this case is an elliptic one.

4.1. The case of a homogeneous function $\theta(\xi)$ in (4.1)

The case $\theta(\xi) = \theta(\xi')$, $\theta \in C^q(S^{n-1})$ was considered in [14]. The operators $B^\alpha_{\theta,\gamma}$ with the symbol (4.1) were realized in [14] as the following potentials

$$(B^\alpha_{\theta,\gamma}\varphi)(x) = \int_{R^n} b^\alpha_{\theta,\gamma}(t)\varphi(x - t)dt\ . \tag{4.4}$$

Their kernels have a singularity on the sphere $|t| = \gamma$. This is power singularity of order $\alpha - \frac{n+1}{2}$, if $0 < \Re\alpha < \frac{n+1}{2}$ and logarithmic one, if $\alpha = \frac{n+1}{2}$. The kernel $b^\alpha_{\theta,\gamma}(t)$ for $n - 1 < \Re\alpha < n$ has the form

$$b^\alpha_{\theta,\gamma}(t) = \frac{\Gamma(n-\alpha)|S^{n-2}|}{(2\pi)^n|t|^{n-\alpha}} \int_{-1}^{1} \frac{(1 - |y|^2)^{\frac{n-3}{2}} M_\theta(t', y)}{\left(-i\left(y - \frac{\gamma}{|t|}\right)\right)^{n-\alpha}}dy\ , \tag{4.5}$$

where $M_\theta(t', y)$ are spherical means of $\theta(\xi')$ over $(n-2)$-dimensional sections of the unit sphere by hyperplanes, introduced in [31] and [32] (in these papers the technique of spherical means was applied for the regularization of symbols of generalized Riesz potentials

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with a homogeneous characteristic). In the case $0 < \Re\alpha \le n - 1$ the function $b_{\theta,\gamma}^{\alpha}(t)$ is represented as a sum of the regularized integral (4.5) and the linear combination of Gauss hypergeometric functions.

In the elliptic case, when $\theta(\sigma) \ne 0$, $\sigma \in S^{n-1}$, the inversion of potential $f = (B_{\theta,\gamma}^{\alpha}\varphi)(x)$, $\varphi \in L_p$, $1 \le p < \frac{n}{\Re\alpha}$, was constructed in [14] in the form

$$(G_{\theta,\gamma}^{\alpha}f)(x) = \lim_{\varepsilon \to 0} \left(g_{\theta,\gamma,\varepsilon}^{\alpha} * f \right)(x), \tag{4.6}$$

where

$$g_{\theta,\gamma,\varepsilon}^{\alpha}(t) = \mathcal{F}^{-1}\left[\frac{|\xi|^{\alpha}}{\theta(\xi)} exp\left(-i\gamma|\xi| - \varepsilon|\xi|^2\right) \right](t),$$

the limit in (4.6) being taken in the L_p-norm or almost everywhere.

In non-elliptic cases the most general character of degeneracy of the symbol (4.1), considered in [14], was

$$mes\left\{\xi \in R^n \backslash \{0\} : \theta(\xi') = 0\right\} = 0. \tag{4.7}$$

We denote

$$(U_{\theta,\gamma}^{\alpha}f)(x) = \overset{L_p}{\underset{\varepsilon \to 0}{\lim}} \overset{L_2}{\underset{\delta \to 0}{\lim}} \left(u_{\theta,\gamma,\varepsilon,\delta}^{\alpha} * f \right)(x), \tag{4.8}$$

where

$$u_{\theta,\gamma,\varepsilon,\delta}^{\alpha}(t) = \mathcal{F}^{-1}\left[\frac{\overline{\theta(\xi')}|\xi|^{\alpha}exp\left(-i\gamma|\xi| - \varepsilon|\xi|^2\right)}{|\theta(\xi')|^2 + i\delta)} \right](t) .$$

Theorem 4.1. ([14]) *Let* $\frac{n-1}{2} < \Re\alpha < n$, $\theta \in C^q(S^{n-1})$, *where* $q = 2$ *if* $n = 2$ *and*

$$q = \begin{cases} \frac{3n-1}{2} & n \quad is\ even \\ \\ \frac{3n}{2} + 1 & n \quad is\ odd \end{cases}$$

if $n \ge 3$; $\varphi \in L_p$, $1 \le p < \frac{n}{\Re\alpha}$ *with the additional restriction* $p \le 2$ *for* $\Re\alpha < \frac{n}{2}$ *and the condition (4.7) is fulfilled. Then*

$$U_{\theta,\gamma}^{\alpha}B_{\theta,\gamma}^{\alpha}\varphi = \varphi$$

4.2. The case of radial functions $\theta(\xi)$ in (4.1)

Here we assume that $\theta(\xi) = \theta(|\xi|)$ where

$$\theta(r) \in \Lambda^m(R_+^1) \ := \left\{ f(r) \in C^m(R_+^1) : \ f^{(k)}(r) \le cr^{-k}, \ 0 \le k \le m \right\}.$$

This class contains, in particular, some functions, slowly oscillating at infinity (for example, $f(r) = cos\ ln\ r$). It was shown in [16] that convolution operators with the symbol (4.1) may be represented in the form (4.4) with the kernel

$$b_{\theta,\gamma}^{\alpha}(t) = \frac{(2\pi)^{-\frac{n}{2}}}{|t|^{n-\alpha}} \int\limits_0^{\infty} |y|^{\frac{n}{2}-\alpha} e^{i\gamma\frac{y}{|t|}} \theta\left(\frac{y}{|t|}\right) J_{\frac{n-2}{2}}(y)dy \ , \tag{4.9}$$

$\frac{n+1}{2} < \Re\alpha < n$. In the case $\Re\alpha \le \frac{n+1}{2}$ the integral in the right-hand side of (4.9) is interpreted in the sense of regularization. For Hölder-type functions stabilizing at infinity: $\theta(r) \in C^{m,\beta}(\dot{R}^1_+)$ $(\subset \Lambda^m)$, this class having been introduced in [39], it was shown in [10] that $b^\alpha_{\theta,\gamma}(t)$ has singularities on the sphere $|t| = \gamma$ as in the case of homogeneous functions $\theta(\xi)$ in (4.1). Inversion of potentials $B^\alpha_{\theta,\gamma}\varphi$ with L_p- densities was constructed in [16] both in elliptic $(\inf_{r>0} |a(r)| \ne 0)$ and general non-elliptic $(mes \{r > 0 : \theta(r) = 0\} = 0)$ cases. The corresponding results are similar to those in the Subsection 4.1 and we do not dwell on their explicit formulation.

We consider here one important non-elliptic case - the so-called quasi-elliptic case-when $a(r) \ne 0$, $r > 0$ and one or both limits $\lim_{r\to 0} a(r)$, $\lim_{r\to\infty} a(r)$ equal zero. In this case it is possible to construct the inversion of potential $B^\alpha_{\theta,\gamma}$ more effectively in comparison with the general non-elliptic situation. We assume that

$$a(r) \in \Lambda^m(R^1_+), \ m > [\tfrac{n}{2}]; \ a(r) \ne 0, \ r \in R^1_+;$$

$$|a(r)|^{-1}exp\left\{-\eta r - \tfrac{\delta}{r}\right\} \le c \quad for \ some \quad \eta, \ \delta \ge 0. \tag{4.10}$$

Theorem 4.2. ([16]) *Let* $\frac{n-1}{2} < \Re\alpha < n$, $\varphi \in L_p$, $1 \le p < \frac{n}{\Re\alpha}$ *and* $\theta(r)$ *satisfy the condition (4.10). Then*

$$N^\alpha_{\theta,\gamma}B^\alpha_{\theta,\gamma}\varphi = \varphi,$$

where

$$(N^\alpha_{\theta,\gamma}f)(x) = \lim_{\varepsilon\to 0}\left(n^\alpha_{\theta,\gamma,\varepsilon} * f\right)(x), \tag{4.11}$$

$$n^\alpha_{\theta,\gamma,\varepsilon}(t) = \mathcal{F}^{-1}\left[\frac{|\xi|^\alpha}{\theta(\xi)}exp\left(-i\gamma|\xi| - \varepsilon|\xi|^2\right) - \frac{\varepsilon}{|\xi|^{2n}}\right)\right](t),$$

the limit in (4.11) being taken in the L_p-norm for $1 < p < \frac{n}{\Re\alpha}$, or almost everywhere if $1 \le p < \frac{n}{\Re\alpha}$.

4.3. Inversion of the operator H^α defined in (2.3) and (2.5)

Within the framework of AIO's method the operator inverse to the operator H^α is constructed in the form

$$R^\alpha f = \lim_{\varepsilon\to 0}\mathcal{F}^{-1}\left(\frac{exp\left(-\varepsilon|\xi|^2\right)}{h_\alpha(|\xi|)}\right) * f, \tag{4.12}$$

where $h_\alpha(|\xi|)$ is the symbol (3.4).

Theorem 4.3. ([17]) *Let $0 < \alpha < 2$, $\varphi \in L_p$, $1 \le p < \frac{n}{n+\alpha-2}$. Then in the elliptic case*

$$R^\alpha H^\alpha\varphi = \varphi, \tag{4.13}$$

the limit in (4.12) being taken in L_p-norm or almost everywhere.

The inversion formula is also valid for

$$\varphi \in L_p, \quad \frac{4-\alpha-2n}{2(1-n)} < \frac{1}{p} < \frac{(n-2)(\alpha+2n)+4}{2n(n-1)}$$

(for such p the operator H^α may be extended to a bounded operator from L_p into L_q in accordance with the Theorem 3.5).

The range $H^\alpha(L_p)$ was also described in [17] in terms of the inverse operator (4.12).

4.4. Inversion of the operator M^α defined in (4.2) and more general convolution operator of Strichartz-Peral-Miyachi-type

Here we consider the operator (4.2) as well as more general operator $M^{\alpha,\beta}$ with the symbol

$$\mu_{\alpha,\beta}(|\xi|) = \Gamma(\frac{n}{2} + \alpha)(1 + |\xi|^2)^{-\frac{\beta}{2}} \left(\frac{|\xi|}{2}\right)^{1-\frac{n}{2}-\alpha} J_{\frac{n}{2}+\alpha-1}(|\xi|) , \qquad (4.14)$$

$$\alpha, \ \beta \ \in R^1 \ (M^\alpha = M^{\alpha,0}, \ \alpha > 0).$$

These operators play an important role in different problems of analysis and mathematical physics (see, for example [12]-[13],[26]-[27],[40]-[42]). We base ourselves on the following result.

Theorem 4.4. *Let $M_{p,q}$ denote the class of Fourier $p \to q$-multipliers. Then*

1) $\mu_{\alpha,\beta} \in M_{p,q}$ $1 < p \le q < \infty$ *if and only if* $p \le q$, $\dfrac{1}{p} + \dfrac{1}{q} \le 1$, $\dfrac{1}{p} - \dfrac{n}{q} \le \alpha + \beta$

or $p \le q$, $\dfrac{1}{p} + \dfrac{1}{q} \ge 1$, $\dfrac{n}{p} - \dfrac{1}{q} \le \alpha + \beta + n - 1$;

2) $\mu_{\alpha,\beta} \in M_{1,q}$ $1 \le q < \infty$ *if and only if* $\alpha + \beta > \dfrac{1}{q'}$ $(\dfrac{1}{q} + \dfrac{1}{q'} = 1)$.

This theorem was proved for different values of parameters α and β in [12]-[13],[26]-[27] and [40]. The formulation, given above, is taken from [27].

We consider the general case of the operator $M^{\alpha,\beta}$. Within the framework of AIO's method, the inverse operator to $M^{\alpha,\beta}$ may be constructed in the form

$$L^{\alpha,\beta}f = \lim_{\varepsilon \to 0}^{L_p} \lim_{\delta \to 0}^{L_2} L_{\varepsilon,\delta}^{\alpha,\beta}f, \qquad (4.15)$$

where $L_{\varepsilon,\delta}^{\alpha,\beta}$ is the operator, generated by the following $p-$ multiplier

$$\frac{|\xi|^{\frac{n}{2}+\alpha-1}exp \ (-\varepsilon|\xi|^2)(1+|\xi|^2)^{\frac{\beta}{2}}}{\Gamma(\frac{n}{2}+\alpha)[J_{\frac{n}{2}+\alpha-1}(|\xi|)+i\delta]}.$$

Theorem 4.5. [17]-[18] *Let p, α and β ($1 \le p \le 2$; $\alpha, \beta \in R^1$) be such that the operator $M^{\alpha,\beta}$ is bounded from L_p into L_q for some q in accordance with Theorem 4.4. Then*

$$L^{\alpha,\beta}M^{\alpha,\beta}\varphi = \varphi, \ \varphi \in L_p,$$

where $L^{\alpha,\beta}$ is the operator (4.15); the limit in L_p-norm in (4.15) may be replaced the almost everywhere limit. Moreover,

$$M^{\alpha,\beta}(L_p) = \{f : f \in L_q, \ L^{\alpha,\beta}f \in L_p\} ,$$

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where q, $1 \leq q \leq 2$ is an arbitrary number such, that $M^{\alpha,\beta}$ is bounded from L_p into L_q.

We observe that the proof of embedding

$$\{f : f \in L_q, \ L^{\alpha,\beta}f \in L_p\} \ \subset M^{\alpha,\beta}$$

is essentially based on the possibility of approximation in the $L_{q'}-$ norm $(1 < q \leq 2)$ and in the norm of the space $C_0 = \{f : f \in C(R^n), \ f(\infty) = 0\}$ of a function $\omega \in \mathcal{S}$ by functions ω_N from the Lizorkin-type class Φ_V invariant for the operator $M^{\alpha,\beta}$, where

$$V = \{\bigcup_{i=1}^{\infty} S_i\} \bigcup \{0\} \, , S_i = \{x : |x| = \nu_i\} \, , i = 1, 2,$$

$\{\nu_i\}$ being the sequence of positive real roots of the function $J_{\frac{n}{2}+\alpha-1}(z)$. The class Φ_V and their duals Ψ_V, where V is an arbitrary closed set in R^n were investigated in [30],[33] and [36]. The denseness of Φ_V (*mes V = 0*) in L_p is known (see [33] and [36]) for $2 \leq p < \infty$ (in the case $1 < p < 2$ it was proved in [33] and [36] for special types of sets). Theorem 3.4 provides boundedness of $M^{\alpha,\beta}$ from L_p into L_q, $q \leq 2$ for p, α, β under the consideration.

In the case $\beta = 0$ we obtain the following result for the operator M^{α} defined in (4.2).

Theorem 4.6. [17]-[18] *Let $\alpha > 0$. $1 \leq p \leq 2$, and let L^{α} be the operator (4.15) with $\beta = 0$. Then*

$$L^{\alpha}M^{\alpha}\varphi = \varphi, \ \varphi \in L_p.$$

Moreover,

$$M^{\alpha}(L_p) = \{f : f \in L_q, \ L^{\alpha}f \in L_p\} \, ,$$

where $q \in [1,2]$ is an arbitrary number such that M^{α} is bounded from L_p into L_q.

V. Inversion of some Riesz potentials with oscillating kernels

In the paper [7] the inversion of Riesz potentials

$$(\mathbb{K}_{\theta}^{\alpha}\varphi)(x) = \int_{R^n} \frac{\theta(t)}{|t|^{n-\alpha}}\varphi(x - t)dt \, , \ \ 0 < \alpha < n, \tag{5.1}$$

with the characteristic $\theta(t) = e^{i\gamma|t|}$, $\gamma > 0$ was constructed within the framework of AIO's method. The arising difficulties are similar to those which appear in the inversion problem for acoustic potentials (see [43] and [37]): oscillation of the kernels of the corresponding potentials generates singularities of their symbols on spheres. The most difficult point in the inversion problem for potentials with the oscillating exponent $e^{i\gamma|t|}$ in their characteristics, appears in the non-elliptic case when the symbol of the potential degenerates on some set in R^n and has singularities on the sphere $|\xi| = \gamma$. Another problem arising here, is connected with the fact that denseness in L_p of the Lizorkin-type space Φ_V, in

the case of the sphere $V = \{\xi : |\xi| = \gamma\}$, remains unknown when $1 < p < 2$, this space being invariant with respect to the corresponding potential operator.

This case was considered in [25], where the inversion of potentials $\mathbb{K}^\gamma_{\alpha,a}$ of the form (5.1) with the characteristic

$$\theta(t) = (a \cdot t') exp(i\gamma|t|), \ a \in R^n, \ \gamma > 0$$

was constructed. The symbols $v^\gamma_{\alpha,a}(|\xi|)$ of these potentials have the form

$$v^\gamma_{\alpha,a}(|\xi|) = i\pi^{\frac{n}{2}}(a \cdot \xi)M^\alpha_\gamma(|\xi|^2) \ ,$$

where

$$M^\alpha_\gamma(t) = \frac{i^{\alpha+1}\Gamma(\alpha+1)}{\gamma^{\alpha+1}\Gamma(\frac{n}{2}+1)}F\left(\frac{\alpha+1}{2}, \frac{\alpha+2}{2}; \frac{n}{2}+1; \frac{t}{\gamma^2}\right) \ ,$$

if $t < \gamma^2$ and

$$M^\alpha_\gamma(t) = \frac{2^\alpha}{t^{\frac{\alpha+1}{2}}}\left[\frac{\Gamma(\frac{\alpha+1}{2})}{\Gamma(\frac{n+1-\alpha}{2})}F\left(\frac{\alpha+1}{2}, \frac{\alpha+1-n}{2}; \frac{1}{2}; \frac{\gamma^2}{t}\right)\right.$$

$$\left.+\frac{2i\gamma\Gamma(\frac{\alpha+2}{2})}{\sqrt{t}\Gamma(\frac{n-\alpha}{2})}F\left(\frac{\alpha+2}{2}, \frac{\alpha+2-n}{2}; \frac{3}{2}; \frac{\gamma^2}{t}\right)\right] \ ,$$

if $t > \gamma^2$, and $F(a, b; c; z)$ is the Gauss hypergeometric function. The function $M^\alpha_\gamma(|\xi|^2)$ has singularities on the sphere $|\xi| = \gamma$ (of power or logarithmic type) if $\alpha \geq \frac{n-1}{2}$, and it satisfies the ellipticity conditions

$$\inf_{|\xi|<\gamma} |M^\alpha_\gamma(|\xi|^2)| \neq 0, \quad \inf_{|\xi|>\gamma} |\xi|^{\alpha+1}|M^\alpha_\gamma(|\xi|^2)| \neq 0, \tag{5.2}$$

proved in [25] for $0 < \alpha < 1$ or $n-2 \leq \alpha < n$. Within the framework of AIO's method the inversion of potentials $f = \mathbb{K}^\gamma_{\alpha,a}\varphi$, $\varphi \in L_p$ for such α may be constructed in the form

$$(T^\gamma_{\alpha,a}f)(x) = \lim_{\varepsilon \to 0}\left(t^\gamma_{\alpha,a,\varepsilon} * f\right)(x), \tag{5.3}$$

where

$$t^\gamma_{\alpha,a,\varepsilon}(x) = \mathcal{F}^{-1}\left[\frac{(|\xi|^2 - \gamma^2)^l exp\left(-\varepsilon|\xi|^2\right)}{i\pi^{\frac{n}{2}}M^\alpha_\gamma(|\xi|^2)(a \cdot \xi) + i\varepsilon)(|\xi|^2 + \gamma^2(\varepsilon + i)^2)^l}\right](x), \tag{5.4}$$

$l > n - 1 + \frac{n(n+1)}{2}$.

Theorem 5.1. [25] *Let* $0 < \alpha < 1$ *or* $n-2 \leq \alpha < n$, $\varphi \in L_p$, $1 < p < \min\{\frac{2n}{n-1}, \frac{n}{\alpha}\}$. *Then*

$$T^\gamma_{\alpha,a}\mathbb{K}^\gamma_{\alpha,a}\varphi = \varphi, \ \varphi \in L_p \ ,$$

where $T^\gamma_{\alpha,a}$ *is the operator (5.3), the limit in (5.3) is taken in the* $L_p + L_r$-*norm,* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, $\frac{1}{q} < \frac{n+1}{2n}$, *or almost everywhere.*

In the case $1 \leq \alpha < n - 2$ $(n \geq 4)$ when the question about zeros of the function $M_\gamma^\alpha(|\xi|^2)$ remains open, the inversion of the potential $f = \mathbb{K}_{\alpha,a}^\gamma \varphi$, $\varphi \in L_p$, $1 < p < \frac{n}{\alpha}$, $p \leq 2$, may be constructed in the form

$$\mathfrak{T}_{\alpha,a}^\gamma f = \overset{L_p}{\underset{\varepsilon \to 0}{\lim}} \, \overset{L_2}{\underset{\delta \to 0}{\lim}} \, \mathfrak{T}_{\alpha,a,\varepsilon,\delta}^\gamma f, \tag{5.5}$$

where

$$\mathfrak{T}_{\alpha,a,\varepsilon,\delta}^\gamma f == \mathcal{F}^{-1}\left[\frac{\overline{v_{\alpha,a}^\gamma(\xi)}(|\xi|^2 - \gamma^2)^l exp\,(-\varepsilon|\xi|^2)}{(|v_{\alpha,a}^\gamma(\xi)|^2 + i\delta)(|\xi|^2 + \gamma^2(\varepsilon + i)^2)^l} \right] * f.$$

VI. Some open questions

6.1. As is seen from comments to the results of Subsection 4.4 and Section 5, an important role in the justification of the results on the inversion of potential type operators in L_p-spaces is played by the Lizorkin-type spaces Φ_V. As mentioned above, the denseness of Φ_V in L_p for an arbitrary closed set V in R^n of measure zero remains unknown in the case $1 < p < 2$.

6.2. It is of special interest to construct the complete $\mathcal{L}$-characteristic of the operator (3.1). The construction, given in Subsection 3.1, has some gaps.

6.3. There are also open problems in investigation of the convolution operator $B_{\theta,\gamma}^\alpha$ with the symbol (4.1) when $\theta(x)$ belongs to the class

$$\Lambda^m(R^n) = \left\{ \theta(x) : \theta(x) \in C^m(R^n \backslash \{0\}),\ |\mathcal{D}^j \theta(x)| \leq c|x|^{-j},\ 0 \leq j \leq m \right\},$$

which is widest possible, in a natural sense. This class contains the classes considered in Subsections 4.1, 4.2 as well the classes $H_q^m(R^n \backslash \{0\})$ and $C_q^m(R^n \backslash \{0\})$, introduced in [11] and [43], respectively, of functions $\theta(x) = \theta(r\sigma)$, $r \in (0, \infty)$, $\sigma \in S^{n-1}$, differentiable up to the order m with respect to the radial variable r and up to the order q with respect to the spherical variable σ. In particular, the following points are of interest:

a) To obtain an integral representation of the form (4.4) for such an integral operator and investigate properties of the corresponding kernel $b_{\theta,\gamma}^\alpha$, such as the nature of singularities on the sphere $|t| = \gamma$, behavior at infinity and so on. The arising difficulties are caused by the fact that multiplication by the exponent $e^{i\gamma|t|}$ does not preserve the class Λ^m;

b) To construct the $\mathcal{L}$-characteristic of the operator $B_{\theta,\gamma}^\alpha$;

c) To describe the ranges $B_{\theta,\gamma}^\alpha(L_p)$ in non-elliptic cases. This problem is closely connected with denseness in L_p of Lisorkin-type spaces Φ_V (see 6.1), where $V = \{\xi : \theta(\xi) = 0\}$.

Besides the fact that the kernel of the operator $B_{\theta,\gamma}^\alpha$ has singularities on the sphere, our interest to the study of this operator also is stimulated by the following reason. The operator $B_{\theta,\gamma}^\alpha$ strongly converges in L_p as $\gamma \to 0$ to the operator with the symbol $m_{\theta,0}^\alpha(\xi) = \frac{\theta(\xi)}{|\xi|^\alpha}$. It may be shown that the class of convolution operators with symbols

$m_{\theta,0}^{\alpha}(\xi)$, $\theta(r) \in \Lambda^{\infty}$, coincides with the class of Riesz potentials with characteristics in Λ^{∞} (see the surveys [35] and [21] for some aspects of the theory of such potentials). Thus, the class $\{B_{\theta,\gamma}^{\alpha}\}_{\gamma \geq 0}$ contains the above mentioned Riesz potentials as the limiting case.

6.4. It is of special interest to investigate complex powers of second order non-homogeneous differential operators of the form (1.1) but with the Dalambert operator instead of the Laplace one.

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