

Completely monotonic functions

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Abstract

In this expository article we survey some properties of completely monotonic functions and give various examples, including some famous special functions.

Such function are useful, for example, in probability theory. It is known, [3], p.450, for example, that a function w is the Laplace transform of an infinitely divisible probability distribution on $(0, \infty)$, if and only if $w = e^{-h}$ where the derivative of h is completely monotonic and $h(0+) = 0$.

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1 Definitions and some basic criteria

Definition. A function f with domain $(0, \infty)$ is said to be completely monotonic (c.m.), if it possesses derivatives $f^{(n)}(x)$ for all $n = 0, 1, 2, 3, \dots$ and if

$$(-1)^n f^{(n)}(x) \geq 0 \quad (1.1)$$

for all $x > 0$.

The limit $f^{(n)}(0) = \lim_{x \rightarrow 0+} f^{(n)}(x)$, finite or infinite, exists.

It is known (see [21], p. 161) that a necessary and sufficient condition that $f(x)$ be c.m. is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

where $\alpha(t)$ is non-decreasing and the integral converges for $0 < x < \infty$. Hence we conclude that a non-identically zero c.m. function $f(x)$ cannot vanish for any positive x .

From the definition it follows that if $f(x)$ is c.m. and $f^{(n_0)}(x_0) = 0$ at some point $x_0 \in (0, \infty)$ for some $n_0 = 0, 1, 2, 3, \dots$ then its derivatives of greater order are also equal to zero at this point:

$$f^{(n)}(x_0) = 0, \quad n \geq n_0.$$

The following elementary functions are immediate examples of c.m. functions, which is verified directly:

$$e^{-ax}, \quad \frac{1}{(\lambda + \mu x)^\nu}, \quad \text{and} \quad \ln \left(b + \frac{c}{x} \right) \quad (1.2)$$

where $a \geq 0$, $\lambda \geq 0$, $\mu \geq 0$ and $\nu \geq 0$ with λ and μ not both zero and $b \geq 1$, $c > 0$.

Other examples of elementary functions are

$$e^{\frac{a}{x}}, a > 0, \quad \frac{\ln(1+x)}{x}, \quad (1.3)$$

see Corollary to Theorem 3 and Corollary to Theorem 5, respectively.

Remark 1. A trivial observation is that if $f(x)$ is c.m., then

$$f^{(2m)}(x) \quad \text{and} \quad -f^{(2m+1)}(x)$$

are also c.m. This produces immediately some other examples. For example, from (1.3)

$$\frac{\ln(1+x)}{x^2} - \frac{1}{x(1+x)} \quad \text{is} \quad \text{c.m.}, \quad \text{etc}$$

Theorem 1. *If $f(x)$ and $g(x)$ are c.m., then*

$$af(x) + bg(x) \quad \text{where} \quad a \quad \text{and} \quad b \quad \text{are nonnegative constants} \quad \text{and} \quad f(x)g(x)$$

are also c.m.

Proof. The first is obvious, the second is then easily seen from the Leibniz formula

$$\frac{d^n}{dx^n} [f(x)g(x)] = \sum_{k=0}^n \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x). \quad (1.4)$$

□

Theorem 2. *Let $f(x)$ be c.m. and let $h(x)$ be nonnegative with a c.m. derivative. Then $f[h(x)]$ also is c.m.*

Proof. It suffices to refer to the formula for the n -th derivative of a composite function:

$$\frac{d^n}{dx^n} f[h(x)] = \sum_{k=1}^n \frac{1}{k!} f^{(k)}[h(x)] U_k(x) \quad (1.5)$$

where

$$U_k(x) = \sum_{j=0}^{k-1} (-1)^j \binom{k}{j} [h(x)]^j \frac{d^n}{dx^n} [h(x)]^{k-j},$$

see [4], No. 0.430.1. \square

Corollary 1. *Let $f(x)$ and $g(x)$ be c.m. Then*

$$f \left(a + b \int_0^x g(t) dt \right),$$

where a and b are arbitrary nonnegative constants, also is c.m. In particular, the following functions are c.m.

$$f(ax^\alpha + b), \quad a \geq 0, \quad b \geq 0 \quad \text{and} \quad 0 \leq \alpha \leq 1, \quad (1.6)$$

$$f[a + bln(1+x)], \quad a \geq 0, \quad b \geq 0, \quad (1.7)$$

$$f(1 - e^{-x}) \quad (1.8)$$

$$f(Arctg\sqrt{x}) \quad (1.9)$$

if $f(x)$ is c.m.

Corollary 2. *Let $f(x)$ be c.m. and $f(0) < \infty$. Then the functions*

$$\frac{1}{[A - f(x)]^\mu}, \quad A \geq f(0), \quad \mu \geq 0 \quad (1.10)$$

and

$$- \ln \left[1 - \frac{f(x)}{A} \right], \quad A \geq f(0) \quad (1.11)$$

are c.m. From (1.11) it also follows that

$$\frac{f'(x)}{A - f(x)}, \quad A \geq f(0) \quad (1.12)$$

is c.m. since (1.12) reduces to minus the derivative of (1.11).

We note some particular cases of c.m. functions of the type (1.6)-(1.11):

$$e^{-ax^\alpha}, \quad a \geq 0 \quad \text{and} \quad 0 \leq \alpha \leq 1, \quad (1.13)$$

$$\frac{1}{[a + bln(1+x)]^\mu}, \quad a \geq 0, \quad b \geq 0, \quad \mu \geq 0, \quad (1.14)$$

$$\frac{1}{(a - be^{-x})^\mu}, \quad a \geq b > 0, \quad \mu \geq 0, \quad (1.15)$$

$$\ln \left[\frac{x}{x - \ln(1+x)} \right]. \quad (1.16)$$

Another version of the statement for composite functions is given in terms of power series with non-negative coefficients by the following obvious theorem.

Theorem 3. Let $y = f(x)$ be c.m. and let the power series

$$\varphi(y) = \sum_{k=0}^{\infty} a_k y^k$$

converge for all y in the range of the function $y = f(x)$. If $a_k \geq 0$ for all $k = 0, 1, 2, \dots$. Then

$$\varphi[f(x)]$$

is c.m.

Corollary. If $f(x)$ is c.m., then

$$e^{f(x)}$$

is c.m. In particular, the functions

$$e^{ax^\alpha}, \quad a \geq 0, \quad \alpha \leq 0, \quad (1.17)$$

$$(1+x)^{\frac{a}{x}} = e^{a \frac{\ln(1+x)}{x}}, \quad a \geq 0, \quad (1.18)$$

and

$$\left(a + \frac{b}{x}\right)^\mu = e^{\mu \ln(a + \frac{b}{x})}, \quad a \geq 1, b > 0, \quad \mu \geq 0 \quad (1.19)$$

are c.m.

Below in Lemma 1 we shall extend the statement on complete monotonicity of the function $(a + \frac{b}{x})^\mu$ for all $a \geq 0$.

Remark 2. Observe that, besides the function e^x , for example, the functions

$$\sinh x, \quad \cosh x, \quad L_n^{(\alpha)}(-x), \quad I_n(x), \quad E_{\alpha,\beta}(x) \quad (1.20)$$

have nonnegative coefficients in their power series expansion; $L_n^{(\alpha)}(-x)$ is the Laguerre polynomials, $I_n(x)$ is the modified Bessel function of the first kind and $E_{\alpha,\beta}(x)$ is the generalized Mittag-Leffler function, see (4.2).

2 Some special functions and integral transforms as c.m. functions

1. A theorem on c.m. functions representable by integrals.

A natural idea is to pass from series representation with positive coefficients, as in Theorem 3, to integral transforms with non-negative densities. Let

$$F(x) = \int_c^d K(x, t) f(t) dt, \quad 0 \leq c < d \leq \infty. \quad (2.21)$$

Obviously, if $K(x, t)$ is c.m. in x for all $t \in (0, \infty)$ and $f(t)$ is nonnegative, then formal differentiation shows that $F(x)$ also is c.m. Since the differentiation under the integral

sign is possible in case of uniform convergence of the differentiated integral, we arrive at the following statement.

Theorem 4. *Let $K(x, t)$ be c.m. in x for all $t \in (0, \infty)$ and let a nonnegative locally integrable function $f(t)$ be such that all the integrals*

$$\int_c^d \frac{\partial^n}{\partial x^n} K(x, t) f(t) dt, \quad n = 0, 1, 2, 3, \dots \quad (2.22)$$

converge uniformly in a neighborhood of any point $x \in (0, \infty)$. Then $F(x)$ is c.m.

First of all we show that by means of the above theorem we may prove the following auxiliary result.

Lemma 1. *The function*

$$\left(a + \frac{b}{x} \right)^\mu, \quad a \geq 0, \quad b > 0, \quad \mu \geq 0, \quad (2.23)$$

is c.m.

Proof. The function $a + \frac{b}{x}$ is obviously c.m. Thus by Theorem 1, $\left(a + \frac{b}{x} \right)^m$ is c.m. in case m is an integer. Therefore, it remains but to prove the lemma for the case when $0 < \mu < 1$. The representation

$$\left(1 + \frac{1}{x} \right)^\mu = 1 + \frac{\mu}{x^\mu} \int_1^\infty \frac{dt}{t^{\mu+1}(xt+1)^{1-\mu}},$$

is valid which may be verified directly by means of the change of variable $t = \frac{1}{xs}$. Since $\mu \leq 1$, the function $\frac{1}{(xt+1)^{1-\mu}}$ is c.m. Then the result follows by Theorem 4. \square

Corollary. *The function*

$$\left(a + \frac{b}{x^\alpha} \right)^\mu, \quad a \geq 0, \quad b \geq 0, \quad \mu \geq 0, \quad 0 \leq \alpha \leq 1, \quad (2.24)$$

is c.m.

2. Complete monotonicity of some special functions.

By means of Theorem 4 we obtain the following statement on complete monotonicity of some special functions.

Theorem 5. *The following special functions:*

the confluent hypergeometric function (Kummer's function) ${}_1F_1(a, c; -x)$, $c > a > 0$,
the Gauss hypergeometric function ${}_2F_1(a, b; c; -x)$, $c > b > 0$, $a > 0$,
the function $x^{\min\{\nu, \frac{1}{2}\}} e^x K_\nu(x)$, where $K_\nu(x)$ is the modified Bessel function (MacDonald's function),
the function $J_\nu^2(x) + Y_\nu^2(x)$, where $J_\nu(x)$ and $Y_\nu(x)$ are the Bessel functions of the first and second kind,
the function $e^{\frac{x^2}{4}} D_\mu(x)$, where $D_\mu(x)$ is the parabolic cylinder function, $\mu < 0$,
the function $x^{-\beta - \frac{1}{2}} e^{\frac{x}{2}} W_{\alpha, \beta}(x)$, where $W_{\alpha, \beta}(x)$ is the Whittaker function, $\alpha < \beta + \frac{1}{2}$,
are c.m.

Proof. The statement of Theorem 5 follows from Theorem 4 and the well known integral representations of the special functions:

$${}_1F_1(a, c; -x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{-xt} t^{a-1} (1-t)^{c-a-1} dt, \quad c > a > 0, \quad [14], p.274; \quad (2.25)$$

$${}_2F_1(a, b; c; -x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1+tx)^{-a} dt, \quad c > b > 0, \quad [14], p.54; \quad (2.26)$$

$$J_\nu^2(x) + Y_\nu^2(x) = \frac{8}{\pi^2} \int_0^\infty K_0(2x \sinh t) \cosh(2\nu t) dt \quad [14], p.93; \quad (2.27)$$

$$D_\mu(x) = \frac{e^{-\frac{x^2}{4}}}{\Gamma(-\mu)} \int_0^\infty e^{-xt} e^{-\frac{t^2}{2}} t^{-\mu-1} dt, \quad \mu < 0, \quad [14], p.328; \quad (2.28)$$

$$W_{\alpha, \beta}(x) = \frac{x^{\beta+\frac{1}{2}} e^{-\frac{x}{2}}}{\Gamma(\frac{1}{2} + \beta - \alpha)} \int_0^\infty e^{-xt} t^{\beta-\alpha-\frac{1}{2}} (1+t)^{\beta+\alpha-\frac{1}{2}} dt, \quad \beta > \alpha - \frac{1}{2}, \quad [14], p.313. \quad (2.29)$$

As regards the MacDonald function, from its well known integral representation

$$e^x K_\nu(x) = \int_0^\infty e^{-x(\cosh t - 1)} \cosh(\nu t) dt \quad x > 0, \quad [14], p.85; \quad (2.30)$$

we immediately conclude that $e^x K_\nu(x)$ is c.m. To state more, that is, that $x^{\min\{\nu, \frac{1}{2}\}} e^x K_\nu(x)$ is c.m., we use another representation:

$$e^x K_\nu(x) = \frac{\sqrt{\frac{\pi}{2x}}}{\Gamma(\nu + \frac{1}{2})} \int_0^\infty e^{-t} t^{\nu - \frac{1}{2}} \left(1 + \frac{t}{2x}\right)^{\nu - \frac{1}{2}} dt \quad x > 0, \quad [4], \text{No. 8.432.8}, \quad (2.31)$$

which is valid for $\nu > -\frac{1}{2}$. By Lemma 1, the factor $\left(1 + \frac{t}{2x}\right)^{\nu - \frac{1}{2}}$ in (2.31) is c.m. with respect to x , if $\nu \geq \frac{1}{2}$. Let $\nu \leq \frac{1}{2}$. We make the change of variables $t = xs$ in the integral in (2.31) and taking also into account that $K_\nu(x) = K_{-\nu}(x)$, arrive at the representation

$$e^x K_\nu(x) = \frac{\sqrt{\frac{\pi}{2}}}{\Gamma(-\nu + \frac{1}{2})} x^{-\nu} \int_0^\infty e^{-xs} s^{-\nu - \frac{1}{2}} \left(1 + \frac{s}{2}\right)^{-\nu - \frac{1}{2}} ds, \quad 0 \leq \nu \leq \frac{1}{2}. \quad (2.32)$$

Since e^{-xs} is c.m. in x , we conclude that $x^\nu e^x K_\nu(x)$ is c.m., $0 \leq \nu \leq \frac{1}{2}$. \square

Remark 3. One may also obtain the complete monotonicity of most of the special functions listed in Theorem 5, not referring to Theorem 4, but directly, making use of the formulas for their n -th derivatives:

$$(-1)^n \frac{d^n}{dx^n} {}_1F_1(a, c; -x) = \frac{\Gamma(a+n)\Gamma(c)}{\Gamma(c+n)\Gamma(a)} {}_1F_1(a+n, c+n; -x) \geq 0, \quad [14], p.264;$$

$$(-1)^n \frac{d^n}{dx^n} {}_2F_1(a, b; c; -x) = \frac{\Gamma(c)\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} {}_2F_1(a+n, b+n; c+n; -x) \geq 0, \quad [14], p.41;$$

$$(-1)^n \frac{d^n}{dx^n} \left[e^{\frac{x^2}{4}} D_\mu(x) \right] = \frac{\Gamma(n-\mu)}{\Gamma(-\mu)} e^{\frac{x^2}{4}} D_{\mu-n}(x) \geq 0, \quad [14], p.326;$$

$$(-1)^n \frac{d^n}{dx^n} \left[e^{\frac{x}{2}} x^{-\beta-\frac{1}{2}} W_{\alpha,\beta}(x) \right] = \frac{\Gamma(\beta-\alpha+n+\frac{1}{2})}{\Gamma(\beta-\alpha+\frac{1}{2})} e^{\frac{x}{2}} x^{-\beta-\frac{n}{2}-\frac{1}{2}} W_{\alpha-\frac{n}{2},\beta+\frac{n}{2}}(x) \geq 0, \quad [14], p.301.$$

In the following Corollary to Theorem 5 we use the error function

$$Erf x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

and the modified incomplete Gamma function

$$\gamma^*(\lambda, x) = \frac{1}{\Gamma(\lambda)x^\lambda} \int_0^x t^{\lambda-1} e^{-t} dt, \quad \lambda > 0.$$

Corollary to Theorem 5. *The functions*

$$\frac{Erf \sqrt{x}}{\sqrt{x}}, \quad \gamma^*(\lambda, x), \quad \lambda > 0 \quad \text{and} \quad \frac{\ln(1+x)}{x}$$

are c.m.

Proof. Indeed, it suffices to observe that

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) = \frac{1}{2} \sqrt{\frac{\pi}{x}} Erf \sqrt{x},$$

$${}_1F_1(\lambda, \lambda+1; -x) = \Gamma(\lambda+1) \gamma^*(\lambda, x), \quad \lambda > 0$$

and

$${}_2F_1(1, 1; 2; -x) = \frac{\ln(1+x)}{x}.$$

□

Remark 4. Making use of the recursion formulas for the MacDonald function $K_\nu(x)$

$$K'_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x} K_\nu(x) \quad (2.33)$$

and

$$x [K_{\nu+1}(x) - K_{\nu-1}(x)] = 2\nu K_\nu(x), \quad (2.34)$$

[4], Nos. 8.486.12 and 8.486.10, one may easily derive some consequences from the fact that $K_\nu(x)$ and even $e^x K_\nu(x)$ are c.m. Thus, from (2.34) we conclude that

$$xe^x [K_{\nu+1}(x) - K_{\nu-1}(x)] \quad \text{is} \quad \text{c.m.}$$

and differentiating the product $e^x K_\nu(x)$, by means of (2.33) we easily obtain that

$$e^x \left[K_{\nu-1}(x) + \left(\frac{\nu}{x} - 1 \right) K_\nu(x) \right] \quad \text{is} \quad \text{c.m.} \quad \text{etc}$$

We observe also that one may make a conclusion on the complete monotonicity of another famous special function, the Riemann zeta-function, as is stated in the next theorem.

Theorem 6. *The generalized Riemann zeta-function*

$$\zeta(x+1, \nu+1) = \sum_{k=0}^{\infty} \frac{1}{(\nu+1+k)^{x+1}} \quad (2.35)$$

is c.m. both in x and ν .

The proof is obtained by the direct differentiation under the series sign, which is obviously possible.

3. Integral transforms of nonnegative functions as complete monotonic functions

Theorem 7. *The following integral transforms are c.m. under the corresponding convergence conditions, whatever the nonnegative function $f(t)$ is:*

1. *The Laplace transforms*

$$F(x) = \int_0^\infty e^{-xt} f(t) dt; \quad (2.36)$$

2. *The Stieltjes-type transforms*

$$F(x) = \int_0^\infty \frac{f(t) dt}{(x+t)^\rho} \quad (2.37)$$

and also the transformation which is close in a sense:

$$F(x) = \int_0^\infty \frac{t^{\gamma-1}}{(x^\alpha + t^\alpha)^a} f(t) dt \quad (a > 0, 0 < \alpha \leq 1, \gamma > 0);$$

3. *The Lambert transforms* ([22], p. 192)

$$F(x) = \int_0^\infty \frac{f(t) dt}{e^{xt} - 1} \quad (2.38)$$

or more generally

$$F(x) = \int_0^\infty \frac{f(t) dt}{(e^{xt} - 1)^m}, \quad m = 1, 2, 3, 4, \dots;$$

4. The Hankel-type transforms

$$F(x) = \int_0^\infty \sqrt{xt} K_\nu(xt) f(t) dt, \quad \nu \geq \frac{1}{2}. \quad (2.39)$$

Proof. The statement of the theorem follows from Theorem 4, since all the kernels in 1.-4. are c.m. Indeed, for 1. and 2., we refer to (1.2), for 3. the complete monotonicity of the kernel $\frac{1}{e^x - 1}$ was stated in (1.15) and the complete monotonicity of the kernel $\sqrt{x} K_\nu(x)$ in 4. is given by Theorem 5. \square

4. The case of integral transforms with kernels homogeneous of degree -1 .

The following theorem gives some useful criterion of monotonicity of integral transforms, which is a consequence of Theorem 4. Let

$$Kf(x) = \frac{1}{x} \int_0^\infty k\left(\frac{x}{t}\right) f(t) dt \quad \text{or} \quad Kf(x) = \frac{1}{x} \int_0^x k\left(\frac{x}{t}\right) f(t) dt \quad (2.40)$$

be an integral transformation with kernel homogeneous of degree -1 .

Theorem 8. *Integral transform (2.40) with a nonnegative kernel $k(t)$ of a c.m. function $f(t)$ also is c.m., under the corresponding convergence conditions.*

Proof. Indeed, we have

$$Kf(x) = \int_0^\infty k\left(\frac{1}{t}\right) f(xt) dt \quad (2.41)$$

for the first of the operators in (2.40) with the upper limit ∞ in (2.41) replaced by 1 in case we deal with the second of the operators in (2.40). Since $f(xt)$ in (2.41) is c.m. with respect to x and $k\left(\frac{1}{t}\right)$ is nonnegative, it remains but to apply Theorem 4. \square

Corollary. *The modified fractional integral*

$$f_\alpha^* = \frac{1}{x^\alpha} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad (2.42)$$

and the modified Love transform (see [13] or [19], p. 696, for the Love transformation)

$$\frac{1}{\Gamma(c)x^c} \int_0^x (x-t)^{c-1} {}_2F_1\left(a, b; c; 1 - \frac{t}{x}\right) f(t) dt, \quad c > b > 0, \quad (2.43)$$

of any c.m. function $f(t)$ are c.m. functions as well.

3 Ratios of Bessel functions

Theorem 9. *The following ratios of the MacDonald functions*

$$y_1(x) = \frac{1}{\sqrt{x}} \frac{K_\nu(\sqrt{x})}{K_{\nu+1}(\sqrt{x})}, \quad y_2(x) = \frac{1}{\sqrt{x}} \frac{K_{\nu+1}(\sqrt{x})}{K_\nu(\sqrt{x})} \quad (3.1)$$

and

$$y_3(x) = \frac{1}{\sqrt{x}} \frac{I_\nu(\sqrt{x})}{I_{\nu+1}(\sqrt{x}) + I_\nu(\sqrt{x})}, \quad y_4(x) = \frac{1}{\sqrt{x}} \frac{I_{\nu+1}(\sqrt{x})}{I_{\nu+1}(\sqrt{x}) + I_\nu(\sqrt{x})} \quad (3.2)$$

are c.m.

Proof. Let

$$\begin{aligned} \varphi_1(t) &= \frac{1}{t [J_{\nu+1}^2(\sqrt{t}) + Y_{\nu+1}^2(\sqrt{t})]}, & \varphi_2(t) &= \frac{1}{t [J_\nu^2(\sqrt{t}) + Y_\nu^2(\sqrt{t})]}, \\ \varphi_3(t) &= \frac{J_\nu^2(\sqrt{t})}{\sqrt{t} [J_{\nu+1}^2(\sqrt{t}) + J_\nu^2(\sqrt{t})]} \quad \text{and} \quad \varphi_4(t) &= \frac{J_{\nu+1}^2(\sqrt{t})}{\sqrt{t} [J_{\nu+1}^2(\sqrt{t}) + J_\nu^2(\sqrt{t})]} \end{aligned}$$

and let

$$\Phi_k(x) = \int_0^\infty \frac{\varphi_k(t) dt}{x+t}, \quad k = 1, 2, 3, 4.$$

Then from [5], [6], [10] we have

$$\begin{aligned} y_1(x) &= \frac{2}{\pi^2} \Phi_1(x), & y_2(x) &= \frac{2\nu}{x} + \frac{2}{\pi^2} \Phi_2(x), \\ y_3(x) &= \frac{1}{\pi} \Phi_3(x), & y_4(x) &= \frac{1}{\pi} \Phi_4(x). \end{aligned} \quad (3.3)$$

Theorem 4 completes the proof. \square

Theorem 10. *The ratio*

$$F(x) = \frac{K_\nu(b\sqrt{x})}{K_\nu(a\sqrt{x})} \quad (3.4)$$

with $b \geq a > 0$ is c.m.

Proof. Since $F(x) = e^{-h(x)}$ with $h(x) = -\ln F(x)$, in view of Theorem 2 it suffices to show that $h(x) \geq 0$ and the derivative $h'(x)$ is c.m. The former is obvious because the MacDonald function $K_\nu(x)$ is decreasing and $b \geq a$. As for the latter, the direct calculation of $h'(x)$ with the recursion formula (2.34) taken into account, yields

$$2\sqrt{x}h'(x) = b \frac{K_{\nu-1}(b\sqrt{x})}{K_\nu(b\sqrt{x})} - a \frac{K_{\nu-1}(a\sqrt{x})}{K_\nu(a\sqrt{x})}.$$

Making use of the first of equations (3.3), we obtain

$$h'(x) = \frac{b^2 - a^2}{\pi^2} \int_0^\infty \frac{1}{(a^2x+t)(b^2x+t)} \frac{dt}{J_\nu^2(\sqrt{t}) + Y_\nu^2(\sqrt{t})}$$

which is c.m. by Theorem 4, since $\frac{1}{(a^2x+t)(b^2x+t)}$ is c.m. and $\frac{1}{J_\mu^2(\sqrt{t})+Y_\mu^2(\sqrt{t})}$ is non-negative. \square

Corollary. *The product*

$$\frac{K_\nu(b\sqrt{x})}{K_\nu(a\sqrt{x})} \cdot \frac{K_\mu(b\sqrt{x})}{K_\mu(a\sqrt{x})}, \quad b \geq a > 0$$

is c.m. for all ν and μ . Compare this with the expression

$$\frac{K_\nu(b\sqrt{x})}{K_\nu(a\sqrt{x})} \cdot \frac{K_\mu(a\sqrt{x})}{K_\mu(b\sqrt{x})}, \quad b \geq a > 0, \quad \mu \geq \nu,$$

which was proved to be c.m. in [11].

4 Mittag-Leffler function and its generalizations

The Mittag-Leffler function

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \quad (4.1)$$

and its generalization

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \quad \beta > 0, \quad (4.2)$$

are known to have various applications in analysis, in particular in fractional calculus and in the theory of fractional differential equations, see for example, [15] and [19] and references therein. We are interested in c.m. properties of these functions.

In [17] it was shown that $E_\alpha(-x)$ is c.m. for $0 < \alpha \leq 1$. This was extended to $E_{\alpha,\beta}(-x)$, see [16] and [20], where it was shown that $E_{\alpha,\beta}(-x)$ is c.m. for $0 < \alpha \leq 1$ and $\beta \geq \alpha$.

Another statement on complete monotonicity is the following.

Theorem 11. *The generalized Mittag-Leffler function $E_{\alpha,\beta}\left(\frac{1}{x}\right)$ is c.m. for all $\alpha > 0$, $\beta > 0$.*

Indeed, this follows directly from Theorem 3, since $\frac{1}{x}$ is c.m.

Since some special functions may be expressed in terms of the generalized Mittag-Leffler function, they are automatically c.m. for suitable values of the parameters. We consider examples.

If $M > 0$ is an integer or half integer, $M = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ then we may write

$$E_{\frac{1}{2},M}(-x) = \sum_{j=0}^{\infty} \frac{(-x)^{j-2M}}{\Gamma\left(\frac{j}{2}\right)} - \sum_{j=0}^{2M-1} \frac{(-x)^{j-2M}}{\Gamma\left(\frac{j}{2}\right)}$$

$$= \frac{(-1)^{2M+1}}{x^{2M}} \left[x E_{\frac{1}{2}, \frac{1}{2}}(-x) + \sum_{k=0}^{2M-1} \frac{(-1)^k x^k}{\Gamma(\frac{k}{2})} \right]. \quad (4.3)$$

As regards the function $E_{\frac{1}{2}, \frac{1}{2}}(-x)$, it may be expressed in terms of the error function

$$Erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

We shall show that the formula

$$E_{\frac{1}{2}, \frac{1}{2}}(-x) = \frac{1}{\sqrt{\pi}} - xe^{x^2} Erfc(x) \quad (4.4)$$

is valid. The proof is direct. Indeed, splitting the summation for odd and even k in (4.2), we have

$$E_{\frac{1}{2}, \frac{1}{2}}(-x) = -xe^{x^2} + \frac{1}{\sqrt{\pi}} {}_1F_1\left(1, \frac{1}{2}; x^2\right).$$

Then from the representation

$$e^{-z} {}_1F_1\left(1, \frac{1}{2}; z\right) = {}_1F_1\left(-\frac{1}{2}, \frac{1}{2}; -z\right) = -\sum_{k=0}^{\infty} \frac{(-z)^k}{(2k-1)k!},$$

[14], p. 267, and from the series expansion for the error function

$$e^{-x^2} + \sqrt{\pi}x Erf(x) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+2}}{(2k+1)(k+1)!},$$

[14], p. 350, we arrive at (4.4).

We observe also that $E_{\frac{1}{2}, \frac{1}{2}}(-x)$ is the Laplace transform of

$$\frac{1}{2\sqrt{\pi}} te^{-\frac{t^2}{4}}.$$

Taking $M = 1$ in (4.3), and taking (4.4) into account, we also obtain

$$E_{\frac{1}{2}, 1}(-x) = e^{x^2} Erfc(x). \quad (4.5)$$

Also, for $\beta = n + 1$ a positive integer, we have

$$E_{1, n+1}(-x) = (-1)^n \frac{e^{-x} - e_{n-1}(-x)}{x^n} \quad (4.6)$$

where

$$e_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$$

is the partial sum of the series expansion of e^x and we assume that $e_{-1}(x) \equiv 0$.

Conclusion. *The functions*

$$e^{x^2} \operatorname{Erfc}(x) = E_{\frac{1}{2},1}(-x), \quad (4.7)$$

$$\frac{1}{\sqrt{\pi}} - xe^{x^2} \operatorname{Erfc}(x) = E_{\frac{1}{2},\frac{1}{2}}(-x) \quad (4.8)$$

and

$$(-1)^n \frac{e^{-x} - e_{n-1}(-x)}{x^n} = E_{1,n+1}(-x) \quad (4.9)$$

are c.m.

We should note that the complete monotonicity of the function $e^{x^2} \operatorname{Erfc}(x)$ also follows in view of Theorem 4 directly from its integral representation

$$e^{x^2} \operatorname{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-2xt-t^2} dt. \quad (4.10)$$

Recently, Kilbas and Saigo [12] and [18] introduced a further generalization of the Mittag-Leffler function. Let α and β be positive and let

$$E_{\alpha,\beta,\lambda}(x) = \sum_{m=0}^{\infty} c_m(\alpha, \beta, \lambda)$$

where

$$c_0(\alpha, \beta, \lambda) = 1, \quad c_m(\alpha, \beta, \lambda) = \prod_{k=0}^{m-1} \frac{\Gamma[\alpha(k\beta + \lambda) + 1]}{\Gamma[\alpha(k\beta + \lambda + 1) + 1]}, \quad m \geq 1,$$

and α, β and λ are such that

$$\alpha(k\beta + \lambda) \neq -1, -2, -3, \dots \quad (4.11)$$

In case $\beta = 1$, it reduces to the Mittag-Leffler function:

$$E_{\alpha,1,\lambda}(x) = \Gamma(\alpha\lambda + 1) E_{\alpha,\alpha\lambda+1}(x)$$

provided (4.11) holds. If $\alpha \leq 1$ and $\lambda \geq 0$, then $E_{\alpha,1,\lambda}(-x)$ is c.m.

If $\alpha = 1$, then if $\lambda > -1$,

$$E_{1,\beta,\lambda}(x) = \Gamma(\mu + 1) E_{1,\mu+1} \left(\frac{x}{\beta} \right),$$

where $\mu = \frac{x+1-\beta}{\beta}$. Hence $E_{1,\beta,\lambda}(-x)$ is c.m. if $\lambda \geq \beta - 1$.

If $\lambda > -\frac{1}{\alpha}$, the coefficients c_m are non-negative and by Theorem 3 we see that

$$E_{\alpha,\beta,\lambda} \left(\frac{1}{x} \right)$$

is c.m.

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