Труды Матем. Инст. Стеклов.
156(1980)

Proceedings of the Steklov
Institute of Mathematics
1983, Issue 2

GENERALIZED RIESZ POTENTIALS AND HYPERSINGULAR INTEGRALS WITH HOMOGENEOUS CHARACTERISTICS, THEIR SYMBOLS AND INVERSION

UDC 517.518.23

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ABSTRACT. An apparatus is developed for solving multidimensional integral equations with the kernel of a Riesz potential in R^n. The inverting operator is constructed with the help of so-called hypersingular integrals (HSI's). A connection between generalized Riesz potentials and the corresponding HSI's is sought in terms of the Fourier transforms. A constructive description of the symbol of a potential is given, and for a number of important cases an efficient method for constructing the HSI's is also given. A special apparatus is constructed for regularization of divergent integrals on the sphere. The symbol of an HSI is also calculated, and it is determined in the form of a convolution with a generalized function.

Bibliography: 49 titles.

Introduction

Generalized Riesz potentials are defined to be the following spatial potentials:

$$(K_\theta^\alpha \varphi)(x) = \int_{R^n} \frac{\theta((x-t)/|x-t|)}{|x-t|^{n-\alpha}} \varphi(t)\, dt, \tag{1}$$

with homogeneous "characteristic" $\theta(x/|x|)$. The investigations in the present article deal with the inversion of the potentials $K_\theta^\alpha\varphi$. An apparatus is thereby developed for solving multidimensional integral equations $K_\theta^\alpha\varphi = f$ of the first kind whose kernels have "power" singularities. In the one-dimensional case a fairly complete theory (invertibility, normal solvability, index, cases of solvability in closed form) has already been worked out for equations of the form

$$\int_a^b \frac{M(x,t)}{|x-t|^{1-\alpha}} \varphi(t)\, dt = f(x) \qquad (-\infty \leq a < b \leq \infty) \tag{2}$$

with a function $M(x, t)$ discontinuous at $t = x$ (see [20]–[23], [16]–[19], and [3], §54). The equations $K_\theta^\alpha\varphi = f$ we shall consider here are a certain analogue of them: the numerator is allowed to have a discontinuity at $t = x$ of homogeneous function type. The nature of the multidimensional equations studied here turns out to be significantly richer because of typically multidimensional problems: in particular, the one-dimensional case is poor in homogeneous functions, admitting only those of the form $\theta(x) = c_1 + c_2 \operatorname{sgn} x$, $x \in R^1$; moreover, interest in multidimensional equations is increasing because the potentials (1) are the inverses of partial differential

1980 *Mathematics Subject Classification*. Primary 45A05, 45E05, 42B20, 47G05; Secondary 31C15, 46E35, 46F10.

equations for integral α. On the other hand, it should be pointed out that there are more complete results about the Fredholm property in the one-dimensional case (for $n = 1$ the specific nature of the one-dimensional singular integral equations with which the equations (2) are closely connected is essential in the case where the function $M(x, t)$ is discontinuous).

The question of an efficient representation of the symbol of the potential (1) has turned out to be interesting and meaningful. This representation involves divergent f.p.-integrals* over the sphere with a singularity on an $(n-2)$-dimensional section of the sphere, introduced in §2. Theorems on existence and representation by convergent constructions (regularizations) are established for such f.p.-constructions. The convergent constructions are in terms of the means of traces on planar sections of the sphere of functions defined on the sphere. These means, introduced in §1.5, turn out to be a convenient tool in the study of a number of (typically multidimensional) problems. In particular, we give an application of them to multidimensional singular operators (§1.6). A constructive representation of the symbols of the potentials (1) (see §2.3) involves overcoming considerable difficulties caused, on the one hand, by the use of techniques of f.p.-integrals over the sphere, and, on the other hand, by the "bad" behavior of the potential kernel at infinity.

We construct the operator inverse to the potential (1) in the form of a so-called hypersingular integral (HSI)

$$\int_{R^n} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}} \Omega\left(\frac{t}{|t|}\right) dt \tag{3}$$

(see §4).

Is it possible to explicitly construct a homogeneous function $\Omega(\sigma)$ such that the operator (3) will be inverse to K_θ^α? In the case of a Riesz potential (i.e., $\theta(\sigma) \equiv \text{const}$) this is possible (see [25]), and $\Omega(\sigma) \equiv \text{const}$. In §§5 and 6 we give a, generally speaking, positive answer to this question for an arbitrary sufficiently smooth function $\theta(\sigma)$ in the case where the symbol of the potential (1) is nondegenerate on the unit sphere (elliptic case), and we present a sufficiently efficient construction of the function $\Omega(\sigma)$ for the inverting hypersingular integral (3). We call it the characteristic of the HSI *associated* with the characteristic $\theta(\sigma)$ of the potential. The cases of integral values $\alpha = 1, 2, 3, \ldots$ are to a certain extent exceptional here. They are given special consideration.

Results relating directly to HSI's are given in §4. In particular, the symbols of HSI's are computed and a positive answer (for arbitrary $\alpha > 0$) is given to the question of regarding them as convolutions with the generalized function $\Omega(x')/|x|^{n+\alpha}$. Simultaneously with this we give a result of independent interest asserting that any homogeneous differential operator $P_\alpha(D)$ of order α can be expressed as an HSI with some characteristic that can be explicitly constructed. In proving this result we obtain, in passing, a criterion found in [30] for the harmonicity of polynomials.

In considering the invertibility of the potentials (1) in the framework of L_p-spaces with the help of HSI's the latter are interpreted as limits of truncated HSI's in the

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* *Editor's note.* f.p.-integral stands for finite part *of the* integral—see §2, Definition 5.

L_p-norm. An essential role in the question of convergence of these truncations is played by the assertion (established in §§6.1 and 6.2 and, moreover, of independent interest) that the kernel $k_\theta^\alpha(x) = \theta(x')/|x|^{n+\alpha}$ of the potential (1) serves as a "fundamental" solution of the hypersingular operator whose characteristic is associated with $\theta(x')$. By a result in §4 (on the representation of homogeneous differential operators by HSI's), the formulas obtained contain the fundamental solutions of homogeneous elliptic differential operators.

Some of the results presented here were announced in [27].

NOTATION. R^n is the n-dimensional Euclidean space; $x = (x_1,\dots,x_n)$, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, $x' = x/|x|$; $\mathbf{j} = (1,0,\dots,0)$, $\sigma \cdot x = \sigma_1 \cdot x_1 + \cdots + \sigma_n \cdot x_n$; Σ_{n-1} is the unit sphere in R^n with center at the origin, $|\Sigma_{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$; $Y_{k\mu}$ are the spherical harmonics of order k; $(a)_k = a(a+1)\cdots(a+k-1)$; $[\alpha]$ is the integer part of the number α, and $\{\alpha\}$ is its fractional part; $F\varphi = \hat{\varphi}(x) = \int_{R^n} e^{ix\cdot t}\varphi(t)\,dt$; $\tilde{f}(x) = (2\pi)^{-n}\int_{R^n} e^{-ix\cdot t}f(t)\,dt$; $\|f\|_p = \|f\|_{L_p(R^n)}$; S is the Schwartz class of test functions; and $D = (\partial/\partial x_1,\dots,\partial/\partial x_n)$.

§1. Auxiliary facts and assertions

1. On choice of a rotation smooth with respect to a parameter. Let $\tau = \mathrm{rot}_x t$ be a rotation in R^n carrying $t \in R^n$ into $\tau \in R^n$ in such a way that

$$\frac{x}{|x|} = \mathrm{rot}_x \mathbf{j}, \qquad \mathbf{j} = (1,0,\dots,0). \tag{1.1}$$

The point $x \in R^n\backslash\{0\}$ is called the *parameter* of the rotation $\mathrm{rot}_x t$. For a fixed value of x the choice of the rotation is not uniquely determined by the single condition (1.1) for $n \geqslant 3$, and $\mathrm{rot}_x t$ denotes one of the possible rotations satisfying (1.1). For two different points x the choice of the rotation can be realized arbitrarily, and for a fixed value of t the function $\mathrm{rot}_x t$ is a multi-valued function of the parameter x. Is it possible to determine a rule for choosing the rotations $\mathrm{rot}_x t$ for all $x \in R^n\backslash\{0\}$ according to the condition (1.1) in such a way as to obtain a single-valued function $\mathrm{rot}_x t$ that is infinitely smooth in x (away from the origin)? Such a global choice—a single rule for all $x \in R^n\backslash\{0\}$—is possible in the planar case $n = 2$. But even in the case $n = 3$ every rotation has a one-dimensional set of fixed points (the set of singularities with respect to the variable x), which rules out the desired global choice. We show that for $n \geqslant 3$ it is possible to choose a rotation $\mathrm{rot}_x t$ that is infinitely smooth in x everywhere except on some subspace of dimension $n-2$, for example, the coordinate space $x_k^2 + x_j^2 = 0$, $k \neq j$, $k, j = 1,2,\dots,n$. Such a rule of choice will be determined.

Let A_x be the matrix of the rotation $\mathrm{rot}_x t$, which, by (1.1), has the form

$$A_x = \left\| \begin{matrix} \frac{x_1}{|x|} b_1^1 \cdots b_1^{n-1} \\ \cdots\cdots \\ \frac{x_n}{|x|} b_n^1 \cdots b_n^{n-1} \end{matrix} \right\|, \qquad (b_1^\nu,\dots,b_n^\nu) = \mathbf{b}^\nu, \quad \nu = 1,2,\dots,n-1.$$

We need to determine an orthonormal set of vectors lying in the hyperplane orthogonal to the vector x in such a way that their components b_i^ν are infinitely differentiable functions of the parameter x everywhere except on an $(n-2)$-dimensional subspace. Let

$$\mathbf{j}^\nu = \{\underbrace{0,\dots,0}_{\nu-1},1,0,\dots,0\}, \qquad \nu = 1,2,\dots,n,$$

so that $\mathbf{j}^1 = \mathbf{j}$. We get the first of the vectors b^ν by projecting one of the unit coordinate vectors $\mathbf{j}^\nu$ onto the plane orthogonal to x. For this, choose a unit vector $\mathbf{j}^{k_1}$ different from $x/|x|$ (in the case when $x/|x|$ turns out to lie on a coordinate axis) and let

$$\mathbf{b}^1 = (r/r_{k_1})\mathbf{j}^{k_1} - (x_{k_1}/r_{k_1})(x/r), \qquad |\mathbf{b}^1| = 1.$$

Here $r = |x|$, $r_{k_1} = \sqrt{r^2 - x_{k_1}^2}$, and an analogous notation will be used in what follows:

$$r_{\alpha\beta} = \sqrt{r^2 - x_\alpha^2 - x_\beta^2}, \qquad r_{\alpha\beta\gamma} = \sqrt{r^2 - x_\alpha^2 - x_\beta^2 - x_\gamma^2}, \quad \text{etc.}$$

It is not difficult to see that $\mathbf{b}^1 \cdot x = 0$ and $|\mathbf{b}^1| = 1$. The subsequent process of constructing an orthogonal system is realized in the form

$$\mathbf{b}^\nu = \frac{r_{k_1k_2\cdots k_{\nu-1}}}{r_{k_1k_2\cdots k_\nu}}\mathbf{j}^{k_\nu} + \sum_{i=1}^{\nu-1} \frac{x_{k_i}x_{k_\nu}}{r_{k_1k_2\cdots k_{\nu-1}}r_{k_1k_2\cdots k_\nu}}\mathbf{j}^{k_i} - \frac{x_{k_\nu}}{r_{k_1k_2\cdots k_{\nu-1}}r_{k_1k_2\cdots k_\nu}}x,$$

where $\nu = 1,\dots,n-1$, and the indices $k_2,\dots,k_n \in \{1,\dots,n\}\setminus\{k_1\}$ are chosen in an arbitrary order. The projections of the vectors $\mathbf{b}^\nu$ on the coordinate axes have the form

$$b_i^\nu = \begin{cases} 0, & i = k_1, k_2, \dots, k_{\nu-1}; \\ r_{k_1k_2\cdots k_\nu} r^{-1}_{k_1k_2\cdots k_{\nu-1}}, & i = k_\nu; \\ -x_{k_\nu}x_i r^{-1}_{k_1k_2\cdots k_{\nu-1}} r^{-1}_{k_1k_2\cdots k_\nu}, & i \neq k_1,\dots,k_\nu. \end{cases} \tag{1.2}$$

Note that the last of the vectors $\mathbf{b}^\nu$ is chosen in the form

$$b_i^{n-1} = \begin{cases} 0, & i = k_1,\dots,k_{n-2}; \\ x_{k_n}\left(x_{k_{n-1}}^2 + x_{k_n}^2\right)^{-1/2}, & i = k_{n-1}; \\ -x_{k_{n-1}}\left(x_{k_{n-1}}^2 + x_{k_n}^2\right)^{-1/2}, & i = k_n \end{cases}$$

(compare with the constructions of Mikhlin in §22 (Chapter IV) of [11], where he discusses the smoothness of a rotation with respect to the angular coordinates of the parameter x, which is always possible to achieve globally). A direct check shows that $|\mathbf{b}^\nu| = 1$ and $\mathbf{b}^\nu\mathbf{b}^\mu = 0$ for $\nu \neq \mu$.

The coefficients of the rotation thus constructed are, in fact, infinitely differentiable with respect to x everywhere except on the subspace $\{x\colon x_{k_{n-1}}^2 + x_{k_n}^2 = 0\}$ of dimension $n-2$.

We write

$$\Pi_{\alpha,\beta} = \{x\colon x_\alpha = x_\beta = 0\}, \qquad \alpha \neq \beta,$$

and introduce the conical neighborhoods

$$V_\varepsilon^{\alpha,\beta} = \{x\colon \rho(x, \Pi_{\alpha,\beta}) < \varepsilon|x|\} = \{x\colon x_\alpha^2 + x_\beta^2 < \varepsilon^2|x|^2\}$$

of the subspaces $\Pi_{\alpha,\beta}$. The intersection of all the cones $V_\varepsilon^{\alpha,\beta}$, $\alpha, \beta = 1,\dots,n$, is empty for sufficiently small ε, $0 < \varepsilon < 2^{1/2}$ (indeed, if $x_\alpha^2 + x_\beta^2 < \varepsilon^2|x|^2$ for all $\alpha \neq \beta$, then by summing we would get that $2|x|^2 < \varepsilon^2|x|^2$).

THEOREM 1. *Let $\varphi(\sigma) \in C^m(\Sigma_{n-1})$. It is possible to cover the space R^n by finitely many cones $V_1,\dots,V_N$ and to construct in R^n a rotation $\mathrm{rot}_x t$ that is infinitely differentiable with respect to x in each of the cones V_i, $i = 1,\dots,N$, in such a way that in the interior of each cone*

$$|D_x^k\varphi(\mathrm{rot}_x\sigma)| \leq c|x|^{-|k|}, \qquad x \in V_i, \quad i = 1,\dots,N, \tag{1.3}$$

where c does not depend on $\sigma \in \Sigma_{n-1}$ or $x \in V_i$, and $|k| \leq m$.

PROOF. Since the intersection of all the cones $V_\varepsilon^{\alpha,\beta}$ is empty for $0 < \varepsilon < 2^{1/2}$, the cones $R^n \setminus V_\varepsilon^{\alpha,\beta}$ cover the whole space R^n. Denote these cones by $V_1,\dots,V_N$ and consider an arbitrary one of them $V_i = R^n \setminus V_\varepsilon^{\alpha_i,\beta_i}$. Choosing $k_{n-1} = \alpha_i$ and $k_n = \beta_i$, we construct a rotation $\mathrm{rot}_x\sigma$ according to the above formulas. Then, taking account of the fact that the coefficients of the rotation matrix are homogeneous and the fact that $x_{\alpha_i}^2 + x_{\beta_i}^2 \geq \varepsilon^2|x|^2$ for $x \in V_i$, we easily get the estimate (1.3), where c depends on ε but not on $x \in V_i$.

2. The mapping $\tau = (\sigma + h)/|\sigma + h|$ on the sphere. Let $h \in R^n$. Let us consider the mapping on the sphere defined by the equality

$$\tau = \sigma_h = \frac{\sigma + h}{|\sigma + h|}, \qquad \sigma \in \Sigma_{n-1}. \tag{1.4}$$

a) *The case* $|h| < 1$. The sphere is mapped onto itself in a one-to-one fashion, and the preimage σ is found by the formula $\sigma = \tau\rho(\tau) - h$, where $\rho(\tau) = |\sigma + h| = h \cdot \tau + \sqrt{(h \cdot \tau)^2 + 1 - |h|^2}$.

b) *The case* $|h| = 1$. The transformation (1.4) maps the sphere $\Sigma_{n-1}\setminus\{h\}$ with its point h deleted onto the hemisphere $h \cdot \tau > 0$, and the preimage σ is found by the formula $\sigma = 2\tau(h \cdot \tau) - h$.

c) *The case* $|h| > 1$. The points τ run through the subset of the sphere

$$\Sigma^h = \left\{\tau\colon |\tau| = 1, |\cos(h,\tau)| \geq \sqrt{1 - 1/|h|^2}\right\}, \tag{1.5}$$

and the mapping (1.4) is not one-to-one: each point $\tau \in \Sigma^h$ has two preimages $\sigma_\pm = \tau\rho_\pm(\tau) - h$, where $\rho_\pm(\tau) = h \cdot \tau \pm \sqrt{(h \cdot \tau)^2 + 1 - |h|^2}$.

REMARK 1. In the case $|h| < 1$ not only is the whole sphere carried onto itself, but even each hemisphere based on a plane passing through the vector h is carried onto itself.

3. Hölder functions on the sphere. Hölder functions on the sphere can be defined directly by writing the Hölder condition at points of the sphere, or in terms of the "translation" (1.4), or locally by projection onto a tangent hypersubspace.

DEFINITION 1. We say that

$$f(\sigma) \in C^\lambda(\Sigma_{n-1}), \qquad 0 < \lambda < 1,$$

if $|f(\sigma) - f(\tau)| \leq A|\sigma - \tau|^\lambda$ for all $\sigma, \tau \in \Sigma_{n-1}$.

DEFINITION 1′. We say that

$$f(\sigma) \in C^\lambda(\Sigma_{n-1}), \qquad 0 < \lambda < 1,$$

if $|f(\sigma) - f(\sigma_h)| \leq A|h|^\lambda$ for all $\sigma \in \Sigma_{n-1}$ and $h \in R^n$, $|h| < 1/2$.

DEFINITION 1″. We say that

$$f(\sigma) \in C_{\cdot\cdot}^{\lambda}(\Sigma_{n-1}), \qquad 0 < \lambda < 1,$$

if the projection $f^*(\bar{x})$ of the function $f(\sigma)$ on the tangent hypersubspace at any point $\sigma_0 \in \Sigma_{n-1}$ is a Hölder function of order λ in some neighborhood of σ_0 in this hypersubspace.

The three definitions, in fact, coincide, and we shall write

$$C^{\lambda}(\Sigma_{n-1}) = C_{\cdot}^{\lambda}(\Sigma_{n-1}) = C_{\cdot\cdot}^{\lambda}(\Sigma_{n-1}).$$

The equivalence of 1 and 1″ is simple. Indeed, suppose that $f(\sigma) \in C^{\lambda}(\Sigma_{n-1})$ in the sense of Definition 1. Since $|\Pr\sigma - \Pr\tau|/|\sigma - \tau| = \sin\gamma \geq c > 0$ in a neighborhood of the point σ_0, the inequality $|f(\sigma) - f(\tau)| \leq A|\sigma - \tau|^{\lambda}$ implies that

$$|f^*(\Pr\sigma) - f^*(\Pr\tau)| \leq cA|\Pr\sigma - \Pr\tau|^{\lambda}.$$

The converse argument is also uncomplicated. It remains to show that 1 and 1′ are equivalent. It is easy to see geometrically that $|\sigma - \sigma_n| \leq |h|$. Therefore,

$$|f(\tau) - f(\sigma)| \leq A|\sigma - \tau|^{\lambda} \Rightarrow |f(\sigma) - f(\sigma_h)| \leq A|h|^{\lambda}.$$

Conversely, let $f(\sigma) \in C_{\cdot}^{\lambda}(\Sigma_{n-1})$. In checking Definition 1 we try for arbitrary σ and τ on the sphere to find a vector h in the form $h = h'|\sigma - \tau|$, $h' \in \Sigma_{n-1}$, such that $\tau = \sigma_h$. It is then necessary to find h' from the equation

$$\frac{\sigma + h'|\sigma - \tau|}{|\sigma + h'|\sigma - \tau||} = \tau,$$

i.e., from the equation $(h' + a)/|h' + a| = \tau$, where $a = \sigma/|\sigma - \tau|$. Here, generally speaking, $|a| > 1$, and h' is on the sphere if and only if (see subsection 2, Case c)) $(a \cdot \tau)^2 \geq |a|^2 - 1$. This condition, i.e., $(\sigma \cdot \tau)^2 \geq 2\sigma \cdot \tau - 1$, is certainly satisfied. Consequently, the desired h' exists and, therefore, $|f(\sigma) - f(\tau)| \leq A|h|^{\lambda} = A|\sigma - \tau|^{\lambda}$.

DEFINITION 2. We say that

$$f(\sigma) \in C^{\lambda}(\Sigma_{n-1}), \qquad \lambda > 0,$$

if $f(x/|x|) \in C^{[\lambda]}(R^n \backslash \{0\})$ and (in case λ is not an integer) the derivatives $g(x) = (D^k f)(x)$ of order $|k| = [\lambda]$ are Hölder functions of order $\lambda - [\lambda]$ on the sphere:

$$|g(\sigma) - g(\tau)| \leq A|\sigma - \tau|^{\lambda - [\lambda]}, \qquad \sigma, \tau \in \Sigma_{n-1}. \tag{1.6}$$

Let $C^0(\Sigma_{n-1}) = C(\Sigma_{n-1})$. The analogous class on the interval $[a, b]$ is denoted by $C^{\lambda}[a, b]$.

DEFINITION 3. We say that

$$f(\sigma) \in C_*^{\lambda}(\Sigma_{n-1}), \qquad \lambda > 0,$$

if the following conditions hold:

1) For λ not an integer $f(\sigma)$ satisfies the conditions of Definition 2 with (1.6) replaced by

$$|g(\sigma) - g(\tau)| \leq A|\sigma - \tau|^{\lambda - [\lambda]} \ln \frac{2}{|\sigma - \tau|}, \qquad \sigma, \tau \in \Sigma_{n-1}. \tag{1.7}$$

2) For λ an integer $f(x/|x|) \in C^{\lambda-1}(R^n \setminus \{0\})$, and the derivatives $g(x) = (D^k f)(x)$ of order $|k| = \lambda - 1$ satisfy on the sphere a Lipschitz condition of the form

$$|g(\sigma) - g(\tau)| \leq A|\sigma - \tau| \ln \frac{2}{|\sigma - \tau|}, \qquad \sigma, \tau \in \Sigma_{n-1}. \tag{1.8}$$

REMARK 2. It can be shown similarly to the preceding that in Definition 3 the Hölder conditions (1.7) and (1.8) can be written in terms of the translation (1.4).

4. Some formulas from the theory of spherical harmonics. Let $Y_m(x') = Y_{m\mu}(x')$, $m = 0, 1, 2, \ldots, \mu = 1, \ldots, d(m)$, be an orthonormal basis of spherical harmonics of order m (see [33] or [34]), where $d(m)$ is the dimension of the space of spherical harmonics of order m:

$$d(m) = (n + 2m - 2)\frac{(n + m - 3)!}{m!\,(n - 2)!} \tag{1.9}$$

($d(m) = 2$ for $n = 2$). We shall use repeatedly the Funk-Hecke formula ([40], p. 20, or [1], §11.4)

$$\int_{\Sigma_{n-1}} Y_m(\sigma) f(x' \cdot \sigma)\, d\sigma = \lambda Y_m(x'), \qquad x' = \frac{x}{|x|}, \tag{1.10}$$

for any function $f(t) \in L_1(-1, 1)$ (in the planar case $n = 2$ it is necessary that $\int_{-1}^{1}(1 - t^2)^{-1/2}|f(t)|\, dt < \infty$), where the constant λ is computed by the formula

$$\lambda = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_{-1}^{1} f(t)(1 - t^2)^{(n-3)/2} H_m(t)\, dt, \tag{1.11}$$

with

$$H_m(t) = \begin{cases} (1/C^m_{m+n-3})C_m^{(n-2)/2}(t), \text{ the Gegenbauer polynomial, for } n \geq 3, \\ \cos(m \arccos t), \text{ the Tchebycheff polynomial, for } n = 2. \end{cases} \tag{1.12}$$

We remark that for all $n \geq 2$

$$H_m(t) = \left(-\frac{1}{2}\right)^m \frac{\Gamma((n-1)/2)}{\Gamma(m + (n-1)/2)} (1 - t^2)^{(3-n)/2} \frac{d^m}{dt^m}(1 - t^2)^{m+(n-3)/2} \tag{1.13}$$

by Rodrigues' formula. Therefore, for differentiable functions $f(t)$ it is possible to write the constant λ in the form

$$\lambda = \frac{\pi^{n/2}}{2^{m-1}\Gamma(m + (n-1)/2)} \int_{-1}^{1} (1 - t^2)^{m+(n-3)/2} f^{(m)}(t)\, dt. \tag{1.14}$$

The familiar formula from mathematical analysis

$$\int_{\Sigma_{n-1}} f(x' \cdot \sigma)\, d\sigma = \frac{2\pi^{(n-1)/2}}{\Gamma((n-1)/2)} \int_{-1}^{1} f(t)(1 - t^2)^{(n-3)/2}\, dt, \qquad n = 2, 3, \ldots \tag{1.15}$$

is a particular case of the Funk-Hecke formula (1.10). We mention other particular cases that will be important for us:

$$\int_{\Sigma_{n-1}} Y_m(\sigma)(-ix'\cdot\sigma)^\alpha d\sigma$$
$$= -\frac{\pi^{n/2-1}}{2^\alpha}\Gamma(1+\alpha)\sin\alpha\pi\frac{\Gamma((m-\alpha)/2)}{\Gamma((m+n+\alpha)/2)}i^m Y_m(x'), \quad (1.16)$$

where Re $\alpha > -1$, and

$$(-iy)^\alpha = |y|^\alpha e^{-i(\alpha\pi/2)\operatorname{sgn} y}, \qquad \alpha \in C^1. \quad (1.17)$$

From this, in particular, we get

$$\int_{\Sigma_{n-1}} Y_m(\sigma)(-ix'\cdot\sigma)^k d\sigma = 0,$$
$$k = 0, 1, \ldots, m-1 \quad \text{and} \quad k = m+1, m+3, m+5, \ldots \quad (1.18)$$

Moreover,

$$\int_{\Sigma_{n-1}} Y_m(\sigma)|x'\cdot\sigma|^\alpha d\sigma$$
$$= \begin{cases} cY_m(x') & \text{if } m \text{ is even and } m \neq \alpha+2, \alpha+4, \ldots, \\ 0 & \text{if } m \text{ is odd or } m = \alpha+2, \alpha+4, \ldots; \end{cases} \quad (1.19)$$

$$\int_{\Sigma_{n-1}} Y_m(\sigma)|x'\cdot\sigma|^\alpha \operatorname{sgn}(x'\cdot\sigma)\, d\sigma$$
$$= \begin{cases} cY_m(x') & \text{if } m \text{ is odd and } m \neq \alpha+2, \alpha+4, \ldots, \\ 0 & \text{if } m \text{ is even or } m = \alpha+2, \alpha+4, \ldots, \end{cases} \quad (1.20)$$

where

$$c = 2^{1-\alpha}\pi^{n/2}\Gamma(1+\alpha)\Gamma^{-1}\left(\frac{m+n+\alpha}{2}\right)\Gamma^{-1}\left(\frac{\alpha-m}{2}+1\right).$$

LEMMA 1. *Suppose that $f(\sigma) = \Sigma a_{k\mu}Y_{k\mu}(\sigma)$ is an expansion of a function on Σ_{n-1} in spherical harmonics. If $f(\sigma) \in C^{2m}(\Sigma_{n-1})$ for $m > (n-1)/2$, then*

$$\Sigma |a_{k\mu}| k^r < \infty \quad (1.21)$$

for $r < 2m - n + 1$. Conversely, if (1.21) holds for $r \geqslant (n-2)/2$, then $f(\sigma) \in C^m(\Sigma_{n-1})$ for $m \leqslant r - (n-2)/2$.

Assertions like Lemma 1 are known for the Sobolev classes $W_2^{(m)}(\Sigma_{n-1})$; (see, for example, [11], Chapter VI, §31), for a certain type of fractional class $W_2^{(\lambda)}(\Sigma_{n-1})$ ([42], p. 6), and for $C^\infty(\Sigma_{n-1})$ ([41], p. 232). Lemma 1, which is formulated for the classes $C^m(\Sigma_{n-1})$, can be proved by standard means ([41], p. 232).

5. Mean traces (on planar sections) of functions on the sphere. For a function $\theta(\sigma)$ defined on the sphere Σ_{n-1} we introduce certain means $M_\theta(x', y)$ over $(n-2)$-dimensional sections. In terms of these means we perform, in §2, a regularization of divergent f.p.-integrals over the sphere with a singularity on a planar section of the sphere.

Let $\sigma \cdot x' = y$, $-1 < y < 1$, $x' = x/|x|$, be the hyperplane cutting the sphere Σ_{n-1} at the height $|y|$ from its center along an $(n-2)$-dimensional sphere orthogonally to the vector x. This $(n-2)$-dimensional sphere with center at the point yx' and radius $\sqrt{1-y^2}$ will be denoted by

$$\Sigma_{n-2}^{x'}\left(yx', \sqrt{1-y^2}\right). \tag{1.22}$$

We introduce the mean of a function $\theta(\sigma)$ defined on Σ_{n-1} over the sphere (1.22). In the planar case $n = 2$ this is the arithmetic mean over two points:

$$M_\theta(x', y) = \frac{\theta(\sigma_+) + \theta(\sigma_-)}{2}, \qquad n = 2, \tag{1.23}$$

where

$$\sigma_\pm = \left(\frac{x_1}{|x|}y \pm \frac{x_2}{|x|}\sqrt{1-y^2}, \frac{x_2}{|x|}y \mp \frac{x_1}{|x|}\sqrt{1-y^2}\right) \tag{1.24}$$

are the points of intersection of Σ_1 with the line passing through the point yx' and perpendicular to the vector x'. In the spatial case $n \geqslant 3$ the means $M_\theta(x', y)$ are given by the integral

$$\begin{aligned} M_\theta(x', y) &= \frac{1}{|\Sigma_{n-2}|(1-y^2)^{(n-2)/2}} \int_{\Sigma_{n-2}^{x'}(yx', \sqrt{1-y^2})} \theta(\sigma)\, dS \\ &= \frac{1}{|\Sigma_{n-2}|} \int_{\Sigma_{n-2}^{x'}(0,1)} \theta\left(yx' + \tau\sqrt{1-y^2}\right) dS_\tau. \end{aligned} \tag{1.25}$$

DEFINITION 4. The mean $M_\theta(x', y)$ is called the *x'-mean* (or the *mean in the direction of the vector x'*) of the function $\theta(\sigma)$ at the height y; the quantity

$$M_\theta(x', 0) = \frac{1}{|\Sigma_{n-2}|} \int_{\Sigma_{n-2}^{x'}(0,1)} \theta(\sigma)\, dS \tag{1.26}$$

is called the *equatorial x'-mean* of the function $\theta(\sigma)$.

The following representation of the means $M_\theta(x', y)$ in terms of a volume integral is valid ($n \geqslant 3$):

$$M_\theta(x', y) = \frac{1}{|\Sigma_{n-2}|} \int_{B^{n-2}(0,1)} \frac{\theta_x(z_+) + \theta_x(z_-)}{\sqrt{1-|\xi|^2}}\, d\xi, \tag{1.27}$$

where

$$\theta_x(\sigma) = \theta(\operatorname{rot}_x \sigma), \tag{1.28}$$

and $z_\pm$ are the following points of Σ_{n-1}:

$$z_\pm = \left(y, \xi\sqrt{1-y^2}, \pm\sqrt{1-|\xi|^2}\sqrt{1-y^2}\right), \qquad \xi \in B^{n-2}(0,1). \tag{1.29}$$

To prove (1.27), perform a change of the variable $\tau \in \Sigma_{n-2}^{x'}(0,1)$, induced by the rotation $\operatorname{rot}_x \sigma$ on Σ_{n-1} in (1.25), and then project the integral over the hemispheres

$\sigma_n > 0$ and $\sigma_n < 0$ onto the $(n-2)$-dimensional ball lying in their base:

$$M_\theta(x', y) = \frac{1}{|\Sigma_{n-2}|} \int_{\Sigma^{\mathbf{j}}_{n-2}(0,1)} \theta_x\left(y\mathbf{j} + \sigma\sqrt{1-y^2}\right) dS_\sigma$$

$$= \frac{1}{|\Sigma_{n-2}|} \left\{ \int_{\sigma_n > 0} \theta_x\left(y, \tilde{\sigma}\sqrt{1-y^2}, \sqrt{1-|\tilde{\sigma}|^2}\sqrt{1-y^2}\right) dS_\sigma \right.$$

$$\left. + \int_{\sigma_n > 0} \theta_x\left(y, \tilde{\sigma}\sqrt{1-y^2}, -\sqrt{1-|\tilde{\sigma}|^2}\sqrt{1-y^2}\right) dS_\sigma \right\}, \qquad (1.30)$$

where $\tilde{\sigma} = (\sigma_2, \ldots, \sigma_{n-1})$, $\sigma = (0, \tilde{\sigma}, \sigma_n)$, $\sigma_n^2 + |\tilde{\sigma}|^2 = 1$, $\sigma = (0, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n)$. Representation (1.27) is obtained from this after projection on the base.

An essential role in our subsequent constructions is played by the values

$$\frac{\partial^k M_\theta(x', 0)}{\partial y^k}, \qquad k = 0, 1, 2, \ldots.$$

LEMMA 2. *If $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda > 0$, then*

$$\frac{\partial^k M_\theta(x', 0)}{\partial y^k} \in C^{\lambda - k}(\Sigma_{n-1}), \qquad k = 0, 1, \ldots, [\lambda].$$

PROOF. In the case $n = 2$ the statement of the lemma follows in an obvious way from (1.21) and (1.22). Suppose that $n \geqslant 3$. From (1.30) we get

$$\frac{\partial^k M_\theta(x', 0)}{\partial y^k} = \frac{1}{|\Sigma_{n-2}|} \int_{\Sigma^{\mathbf{j}}_{n-2}(0,1)} \frac{\partial^k}{\partial y^k} \theta_x\left(y\mathbf{j} + \sigma\sqrt{1-y^2}\right) \Big|_{y=0} dS_\sigma.$$

According to Definition 2, it is necessary to show that

$$\int_{\Sigma^{\mathbf{j}}_{n-2}(0,1)} D_x^m \frac{\partial^k}{\partial y^k} \theta_x\left(y\mathbf{j} + \sigma\sqrt{1-y^2}\right) \Big|_{y=0} dS_\sigma \in C^{\lambda - [\lambda]}(\Sigma_{n-1})$$

for $|m| = [\lambda] - k$. The partial derivatives of the function $\theta(\sigma)$ up to order $|m| + k = [\lambda]$ appear under the integral sign. They are Hölder functions of order $\lambda - [\lambda]$. Therefore, it suffices to show that for $|j| = |m| + k = [\lambda]$

$$\int_{\Sigma^{\mathbf{j}}_{n-2}(0,1)} (D^j\theta)(\operatorname{rot}_x \sigma)\, dS_\sigma \in C^{\lambda - [\lambda]}(\Sigma_{n-1}).$$

Let us make use of the local smoothness of the rotation $\operatorname{rot}_x \sigma$ with respect to the parameter x, established in subsection 1. Breaking up the sphere Σ_{n-1} into finitely many subsets Σ^i_{n-1}, $i = 1, \ldots, N$, by cones according to Theorem 1 and assuming that $\operatorname{rot}_x \sigma$ has been chosen for each of these subsets in such a way that $\operatorname{rot}_x \sigma$ is infinitely differentiable with respect to x, we can easily verify the Hölder property on Σ^i_{n-1}:

$$\int_{\Sigma^{\mathbf{j}}_{n-2}(0,1)} |(D^j\theta)(\operatorname{rot}_x \sigma) - (D^j\theta)(\operatorname{rot}_z \sigma)|\, dS_\sigma$$

$$\leqslant A \int_{\Sigma^{\mathbf{j}}_{n-2}(0,1)} |\operatorname{rot}_x \sigma - \operatorname{rot}_z \sigma|^{\lambda - [\lambda]}\, dS_\sigma \leqslant A_1 |x - z|^{\lambda - [\lambda]}$$

for $x, z \in \Sigma^i_{n-1}$, $i = 1, 2, \ldots, n$ (continuity is checked in the case of integral λ). The lemma is proved. The next result is proved in exactly the same way.

LEMMA 3. *If $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda > 0$, then*

$$D_x^m\left(\frac{\partial}{\partial y}\right)^k M_\theta(x', y) \in C^{\lambda-|m|-k}(\Sigma_{n-1} \times [-a, a]),$$

where $0 < a < 1$, m is a multi-index, and k is an integer such that $|m| + k \leqslant [\lambda]$.

We give explicit formulas for computing $\partial^k M_\theta(x', 0)/\partial y^k$ in the cases $k = 1$ and $k = 2$:

$$\frac{\partial M_\theta(x', 0)}{\partial y} = \frac{1}{|\Sigma_{n-2}|}\int_{\Sigma^{x'}_{n-2}(0,1)} \frac{d\theta(\sigma)}{dx'}\, dS_\sigma \tag{1.31}$$

(the equatorial x'-mean of the derivative $d\theta/dx' = x' \cdot \operatorname{grad}\theta(\sigma)$ in the direction of the vector x') and

$$\frac{\partial^2 M_\theta(x', 0)}{\partial y^2} = \frac{1}{|\Sigma_{n-2}|}\left(\int_{\Sigma^{x'}_{n-2}(0,1)} \frac{d^2\theta(\sigma)}{dx'^2}\, dS_\sigma - \sum_{j=1}^n \int_{\Sigma^{x'}_{n-2}(0,1)} \sigma_j \frac{\partial\theta}{\partial\sigma_j}\, dS_\sigma\right), \tag{1.32}$$

where $d^2\theta(\sigma)/dx'^2 = x' \cdot \operatorname{grad}(x' \cdot \operatorname{grad}\theta)$. The formulas (1.31) and (1.32) are derived from (1.30).

REMARK 3. The derivatives $\partial^k M_\theta(x', y)/\partial y^k$ of the x'-means of the function $\theta(\sigma)$ can be written as a certain combination of the x'-means of the derivatives of $\theta(\sigma)$. Namely,

$$\frac{\partial M_\theta(x', y)}{\partial y} = \frac{1}{1 - y^2}\sum_{k=1}^{n}\left[\frac{x_k}{|x|} M_{\partial\theta/\partial\sigma_k}(x', y) - y M_{\sigma_k(\partial\theta/\partial\sigma_k)}(x', y)\right]. \tag{1.33}$$

Indeed, from (1.25) we have

$$\frac{\partial M_\theta(x', y)}{\partial y} = \frac{1}{|\Sigma_{n-2}|}\int_{\Sigma^{x'}_{n-2}(0,1)} \sum_{k=1}^n \left(\frac{x_k}{|x|} - \frac{\tau_k y}{\sqrt{1 - y^2}}\right)\frac{\partial\theta}{\partial\sigma_k}\left(yx' + \tau\sqrt{1 - y^2}\right) dS_\tau. \tag{1.34}$$

Since

$$x_k|x|^{-1} - \tau_k y(1 - y^2)^{-1/2} = \frac{x_k}{|x|(1 - y^2)} - \frac{y}{1 - y^2}\left(y\frac{x_k}{|x|} + \tau_k\sqrt{1 - y^2}\right),$$

(1.34) implies (1.33), by (1.25).

6. Some applications of x'-means of traces over the sphere. Essential use will be made of the means $M_\theta(x', y)$ in §2.2. Here we indicate some other applications of them. The first of the applications (Lemma 4) will also be used in what follows (§2).

a) *Reduction of integrals over the sphere to one-dimensional integrals.*

LEMMA 4. *Suppose that $\theta(\sigma) \in C(\Sigma_{n-1})$ and $(1 - y^2)^{(n-3)/2} f(y) \in L_1(-1, 1)$. Then*

$$\int_{\Sigma_{n-1}} \theta(\sigma) f(x' \cdot \sigma)\, d\sigma = |\Sigma_{n-2}| \int_{-1}^1 M_\theta(x', y)(1 - y^2)^{(n-3)/2} f(y)\, dy, \tag{1.35}$$

where $|\Sigma_0| = 2$ in the case $n = 2$.

PROOF. Let J be the integral on the left-hand side. Then

$$J = \int_{\Sigma_{n-1}} \theta(\mathrm{rot}_x \sigma) f(\sigma_1)\, d\sigma.$$

Breaking the sphere into the hemispheres $\sigma_n > 0$ and $\sigma_n < 0$, and projecting them onto the ball $B^{n-1}(0, 1)$ lying in their base, we get

$$J = \int_{B^{n-1}(0,1)} \left[\theta_x\left(\tau, \sqrt{1-|\tau|^2}\right) + \theta_x\left(\tau, -\sqrt{1-|\tau|^2}\right)\right] f(\tau_1) \frac{d\tau}{\sqrt{1-|\tau|^2}},$$
$$\tau = (\tau_1, \ldots, \tau_{n-1})$$

(in the notation of (1.28)). Passing here to iterated integration, we have

$$J = \int_{-1}^{1} f(\tau_1)\, d\tau_1$$
$$\times \int_{|\xi|<\sqrt{1-\tau_1^2}} \frac{\theta_x\left(\tau_1, \xi, \sqrt{1-\tau_1^2-|\xi|^2}\right) + \theta_x\left(\tau_1, \xi, -\sqrt{1-\tau_1^2-|\xi|^2}\right)}{\sqrt{1-\tau_1^2-|\xi|^2}}\, d\xi,$$

where $\xi \in R^{n-2}$. The substitution $\xi = (1-\tau_1^2)^{1/2}\xi_{\text{new}}$ gives us

$$J = \int_{-1}^{1} (1-\tau_1^2)^{(n-3)/2} f(\tau_1)\, d\tau_1 \int_{|\xi|<1} \frac{\theta_x(z_+) + \theta_x(z_-)}{\sqrt{1-|\xi|^2}}\, d\xi \tag{1.36}$$

(see the notation in (1.29)). By (1.27), we arrive at the right-hand side of (1.35).

REMARK 4. Formula (1.35) allows us to reduce the computation of surface integrals of the form under discussion to the computation of one-dimensional integrals if the means $M_\theta(x', y)$ of the function $\theta(\sigma)$ over the planar sections perpendicular to the vector x' are known (a certain analogue to the Cavalieri principle!). Formula (1.35) can be regarded as a generalization of the Funk-Hecke formula (1.10)–(1.11). However, to get the Funk-Hecke formula from (1.35) it is necessary to compute the means $M_\theta(x', y)$ for the spherical harmonics $\theta(\sigma) = Y_m(\sigma)$. We have not been able to find them directly (without resorting to the Funk-Hecke formula). Comparing (1.35) with the Funk-Hecke formula, we get the following corollary to Lemma 4 (taking account of the fact that $f(y)$ is arbitrary).

COROLLARY. *The means $M_\theta(x', y)$ of the spherical harmonics $\theta(\sigma) = Y_m(\sigma)$ can be computed by the formula*

$$M_\theta(x', y) = H_m(y) Y_m(x'), \tag{1.37}$$

where the $H_m(y)$ are the polynomials (1.12).

Hence, in particular, the means $M_\theta(x', y)$ of a linear ($m = 1$) function, for example, $\theta(\sigma) = \sigma_j$, $j = 1, 2, \ldots, n$, have the form

$$M_{\sigma_j}(x', y) = y\frac{x_j}{|x|}. \tag{1.38}$$

b) *Application of the equatorial means to singular integrals.* Let us consider the multidimensional singular integral

$$N\varphi = \int_{R^n} \frac{\theta(t')}{|t|^n}\varphi(x-t)\,dt \qquad \left(\int_{\Sigma_{n-1}} \theta(\sigma)\,d\sigma = 0\right). \tag{1.39}$$

Let

$$N(x') = \int_{R^n} \theta(t')|t|^{-n} e^{ix\cdot t}\,dt \tag{1.40}$$

be its symbol. It is known ([11], Chapter IV, §22) that smoothness of the characteristic $\theta(t')$ in the cartesian coordinates of the point t' implies the same smoothness of the symbol $N(x')$ in the angular coordinates. Using the equatorial means $M_\theta(x',0)$ of the characteristic $\theta(\sigma)$ and the local smoothness of the rotation with respect to the parameter (established in subsection 1), we show that

$$\theta(t') \in C^\lambda(\Sigma_{n-1}), \quad \lambda \geqslant 1, \quad \Rightarrow N(x') \in C^{\lambda+1}(\Sigma_{n-1}),$$

i.e., that the symbol $N(x')$ is actually smoother by one order than the characteristic and that smoothness of the symbol can be considered not only with respect to the angular coordinates but also with respect to the cartesian coordinates (a result which is apparently true also for $0 < \lambda < 1$). Using the well-known formula ([11], Chapter IV, §22)

$$N(x') = \int_{\Sigma_{n-1}} \theta(\sigma) \ln \frac{1}{-x'\cdot\sigma}\,d\sigma \tag{1.41}$$

(where $\ln y = \ln|y| + \frac{\pi i}{2} - \frac{\pi i}{2}\operatorname{sgn} y, y \in R^1$), we have

$$N(x') = \int_{\Sigma_{n-1}} [\theta(\sigma) - M_\theta(x',0)] \ln \frac{1}{-x'\cdot\sigma}\,d\sigma + M_\theta(x',0)\int_{\Sigma_{n-1}} \ln\frac{1}{-\sigma_1}\,d\sigma.$$

Differentiating under the integral sign, we easily get that

$$\frac{\partial N}{\partial x_k} = -\int_{\Sigma_{n-1}} \frac{\theta(\sigma) - M_\theta(x',0)}{x\cdot\sigma}\sigma_k\,d\sigma - \frac{x_k}{|x|^2}|\Sigma_{n-1}|M_\theta(x',0) \tag{1.42}$$

(the terms with $\partial M_\theta/\partial x_k$ disappear; it is easy to see that the equality

$$\frac{\partial}{\partial x_k} \ln \frac{1}{-x'\cdot\sigma} = \frac{x_k}{|x|^2} - \frac{\sigma_k}{x\cdot\sigma}$$

holds). We remark that the formula (1.42) for differentiating the symbol can be reduced to the form

$$\frac{\partial N}{\partial x_k} = -\text{p.v.}\int_{\Sigma_{n-1}} \frac{\theta(\sigma)}{x\cdot\sigma}\sigma_k\,d\sigma,$$

where the integral is understood in the sense of the Cauchy principal value (regarding this, see §2.2). It is now more convenient for us to deal with its regularization (1.42). Performing the rotation $\sigma = \operatorname{rot}_x \tau$, in (1.42), we have

$$\frac{\partial N}{\partial x_k} = -\frac{1}{|x|}\int_{\Sigma_{n-1}} \frac{\theta(\operatorname{rot}_x\tau) - M_\theta(x',0)}{\tau_1}(\operatorname{rot}_x\tau)_k\,d\tau - \frac{x_k}{|x|^2}|\Sigma_{n-1}|M_\theta(x',0).$$

Here $M_\theta(x',0) \in C^\lambda(\Sigma_{n-1})$, by Lemma 2. Using the local smoothness of $\mathrm{rot}_x \tau$ with respect to x, we break up Σ_{n-1} by cones into subsets of local smoothness according to Theorem 1 and get that the first term also belongs to $C^\lambda(\Sigma_{n-1})$. Consequently, $\partial N/\partial x_k \in C^\lambda(\Sigma_{n-1})$, which is what was required.

7. On the rate of decrease of the Fourier coefficients of Hölder functions with a weight. We have

LEMMA 5. *Let* $f(x) \in C(0, a)$, $0 < \lambda \leqslant 1$, *and* $0 \leqslant \mu < 1$. *Then as* $N \to \infty$

$$\left| \int_0^a x^{-\mu} f(x) e^{iNx}\, dx \right| \leqslant \frac{c}{N^{\min(\lambda, 1-\mu)}}, \tag{1.43}$$

where $a > 0$, *and* c *does not depend on* N.

PROOF. Let J_N be the integral to be estimated. We have

$$J_N = -\int_{\pi/N}^{\pi/N+a} \left(x - \frac{\pi}{N}\right)^{-\mu} f\left(x - \frac{\pi}{N}\right) e^{iNx}\, dx$$

and, therefore,

$$J_N = \frac{1}{2}\int_{\pi/N}^{a} \left[x^{-\mu} f(x) - \left(x - \frac{\pi}{N}\right)^{-\mu} f\left(x - \frac{\pi}{N}\right)\right] e^{iNx}\, dx$$
$$+ \int_0^{\pi/N} \frac{f(x)e^{iNx}}{2x^\mu}\, dx + \int_{a-\pi/N}^{a} \frac{f(x)e^{iNx}}{2x^\mu}\, dx.$$

The estimate of the second and third terms is clear. The first is majorized by the quantity

$$\frac{1}{2}\int_{\pi/N}^{a} x^{-\mu}\left|f(x) - f\left(x - \frac{\pi}{N}\right)\right| dx + \frac{1}{2}\int_{\pi/N}^{a}\left|f\left(x - \frac{\pi}{N}\right)\right| \cdot \left|x^{-\mu} - \left(x - \frac{\pi}{N}\right)^{-\mu}\right| dx$$
$$\leqslant cN^{-\lambda} + cN^{-1+\mu}\int_\pi^{aN} |x^{-\mu} - (x-\pi)^{-\mu}|\, dx$$
$$\leqslant cN^{-\lambda} + cN^{-1+\mu}, \tag{1.44}$$

which is what was required. (The assertion of the lemma is known for the case $\mu = 0$: see [7], from which we borrowed the device for the proof.)

REMARK 5. An analysis of the proof of Lemma 5 shows that it is preserved if the Hölder property of the function $f(x)$ is replaced by the more general condition

$$|f(x+h) - f(x)| \leqslant c(h/(x+h))^\lambda, \qquad x > 0, h > 0. \tag{1.45}$$

Lemma 5 admits the following generalization:

LEMMA 6. *Suppose that* $f(x, y) \in C^\lambda([0, a] \times [0, a^2])$, $0 < \lambda \leqslant 1$, *and* $0 \leqslant \mu < 1$. *Then as* $N \to \infty$

$$\left| \int_0^a x^{-\mu} f\left(x, \sqrt{x}\right) e^{iNx}\, dx \right| \leqslant \frac{c}{N^{\min(\lambda, 1-\mu)}}, \tag{1.46}$$

where c *does not depend on* N.

The nontrivial point here is in the fact that $f(x, \sqrt{x}) \in C^{\lambda/2}[0, a]$, but the estimate (1.43) is preserved. The proof is analogous to that of Lemma 5 with the single difference that the first term on the left-hand side of (1.44) should be estimated as follows:

$$\int_{\pi/N}^{a}\left|f\left(x,\sqrt{x}\right)-f\left(x-\frac{\pi}{N},\sqrt{x-\frac{\pi}{N}}\right)\right|\frac{dx}{x^{\mu}}\leq c\int_{\pi/N}^{a}x^{-\mu}\left|\sqrt{x}-\sqrt{x-\frac{\pi}{N}}\right|^{\lambda}dx$$
$$=cN^{\mu-1-(\lambda)/2}\int_{\pi}^{aN}x^{-\mu}\left|\sqrt{x}-\sqrt{x-\pi}\right|^{\lambda}dx,$$

with the obvious subsequent estimates.

LEMMA 7. *If* $f(x)\in C^{\lambda}[0,1]$, $\lambda>m-1$, $m=1,2,\dots$, *then*

$$\left|f(x)-\sum_{k=0}^{m-1}\frac{x^k}{k!}f^{(k)}(0)\right|\leq cx^{\min(\lambda,m)}. \tag{1.47}$$

PROOF. For $\lambda\geq m$ the lemma follows in an obvious way from Taylor's formula. Let us therefore assume that $m-1<\lambda<m$. By the definition of the class $C^{\lambda}[0,1]$ we have $-cx^{\lambda-m+1}\leq f^{(m-1)}(x)-f^{(m-1)}(0)\leq cx^{\lambda-m+1}$. Integrating from 0 to x, we have $-c_1x^{\lambda-m+2}\leq f^{(m-2)}(x)-xf^{(m-1)}(0)-f^{(m-2)}(0)\leq c_1x^{\lambda-m+2}$. Integrating in a similar way $m-1$ successive times, we arrive at (1.47).

8. An integral representation. For a function $f(y)$ of a single variable having derivatives up to order m the following formula is valid:

$$\frac{d^{m-k}}{dy^{m-k}}\left[\frac{f(y)-\sum_{j=0}^{m-1}(f^{(j)}(0)y^j/j!)}{y^{k+1}}\right]$$
$$=\frac{1}{k!y^{m+1}}\int_0^y t^{m-k-1}(y-t)^{k-1}(kt-(m-k)(y-t))\left[f^{(m)}(t)-f^{(m)}(y)\right]dt, \tag{1.48}$$

where $k=0,1,\dots,m$.

Let us prove (1.48). Note that to the case $k=m$ corresponds the Taylor formula with remainder in integral form. This formula will be applied to the left-hand side of (1.48), denoted by A. Performing the differentiation d^{m-k}/dy^{m-k} according to Leibniz' formula, we get

$$A=\frac{1}{(m-1)!}\sum_{j=0}^{m-k}(-1)^{m-k-j}C_{m-k}^{j}(1+k)_{m-k-j}y^{j-m-1}\frac{d^j}{dy^j}$$
$$\times\int_0^y f^{(m)}(t)(y-t)^{m-1}dt.$$

Suppose first that $k\neq 0$; then $j\leq m-1$, so that

$$A=(-1)^{m-k}\frac{f^{(m)}(y)}{y}\sum_{j=0}^{m-k}a_j$$
$$+\sum_{j=0}^{m-k}\frac{b_j}{y^{m-j+1}}\int_0^y\left[f^{(m)}(t)-f^{(m)}(y)\right]\frac{dt}{(y-t)^{j+1-m}},$$

where

$$a_j = \frac{(-1)^j C_{m-k}^j}{k!}, \qquad b_j = (-1)^{m-k-j}\frac{m-j}{k!} C_{m-k}^j.$$

Obviously, $\sum_{j=0}^{m-k} a_j = 0$, so that

$$A = \frac{1}{y^2}\int_0^y [f^{(m)}(t) - f^{(m)}(y)] \sum_{j=0}^{m-k} b_j \left(1 - \frac{t}{y}\right)^{m-j-1} dt. \tag{1.49}$$

Further,

$$\sum_{j=0}^{m-k} b_j\left(1-\frac{t}{y}\right)^{m-j-1} = \frac{1}{k!}\left(1-\frac{t}{y}\right)\sum_{j=0}^{m-k}(-1)^k(j+k)C_{m-k}^j\left(1-\frac{t}{y}\right)^{j-1}$$
$$= \frac{(1-t/y)^k}{k!}\left[\frac{ky}{y-t}\sum_{j=0}^{m-k}(-1)^j C_{m-k}^j\left(1-\frac{t}{y}\right)^j \right.$$
$$\left. - \frac{d}{dz}\sum_{j=0}^{m-k}(-1)^j C_{m-k}^j (1-z)^j \Big|_{z=t/y}\right],$$

after which (1.49) becomes (1.48).

Suppose now that $k = 0$. Then, analogously to the preceding,

$$A = -\frac{m}{y^2}\int_0^y f^{(m)}(t)\sum_{j=0}^{m-1}(-1)^{m-j-1}C_{m-1}^j\left(1-\frac{t}{y}\right)^{m-j-1} dt + \frac{f^{(m)}(y)}{y}$$
$$= -\frac{m}{y^2}\int_0^y \left(\frac{t}{y}\right)^{m-1} f^{(m)}(t)\, dt + \frac{f^{(m)}(y)}{y}$$
$$= \frac{m}{y^2}\int_0^y [f^{(m)}(t) - f^{(m)}(y)]\left(\frac{t}{y}\right)^{m-1} dt,$$

which coincides with (1.48) for $k = 0$.

REMARK 6. The representation (1.48) is preserved if m is replaced by $m-1$ without changing k; this follows from the fact that

$$\frac{d^{m-1-k}}{dy^{m-1-k}}\left(\frac{y^{m-1}}{y^{k+1}}\right) = 0.$$

The index k can then be assigned the values $k = 0, 1, \dots, m-1$.

9. On a certain finite sum. An important circumstance in the proof of Theorem 4 on symbols of potentials in §2 will be the fact that we are able to compute the following finite sum:

$$A_{k,j,\nu} \overset{\text{def}}{=} \sum_{\mu=0}^{\min(k,j)} (-1)^\mu \frac{(\nu-\mu)!}{\mu!\,(k-\mu)!\,(j-\mu)!}$$
$$= \frac{(\nu-k)!}{k!\,j!}(\nu-j)(\nu-j-1)\cdots(\nu-j-k-1), \tag{1.50}$$

where ν and j are arbitrary natural numbers, and $k = 0, 1, \ldots, \nu$. In particular,

$$A_{k,j,\nu} = 0 \quad \text{for } \nu \geqslant j \quad \text{and} \quad k \geqslant \nu - j - 1. \tag{1.50'}$$

PROOF OF (1.50). Here it is necessary to distinguish the cases $\nu \geqslant j$ and $\nu \leqslant j - 1$. Suppose first that $\nu \geqslant j$. Then for the values $k = 0, 1, \ldots, \nu$ the sum $A_{k,j,\nu}$ is symmetric in k and j: $A_{k,\nu,j} \equiv A_{j,k,\nu}$. Therefore, it suffices to consider the case $j \leqslant k$. Then (1.50) reduces to the form

$$\sum_{\mu=0}^{j} (-1)^{\mu} C_j^{\mu} (\nu - \mu)(\nu - \mu - 1) \cdots (\nu - j + 1) \cdot k(k-1) \cdots (k - \mu + 1)$$
$$= (\nu - k)(\nu - k - 1) \cdots (\nu - k - j + 1). \tag{1.51}$$

Denoting the left-hand side here by $\sigma_{k,j,\nu}$, we form

$$\sigma_{k,j+1,\nu} = \nu(\nu-1) \cdots (\nu - j) + \sum_{\mu=1}^{j} (-1)^{\mu} C_{j+1}^{\mu} (\nu - \mu)(\nu - \mu - 1) \cdots (\nu - j)$$
$$\cdot k(k-1) \cdots (k - \mu + 1) + (-1)^{j+1} k(k-1) \cdots (k - j).$$

Since $C_{j+1}^{\mu} = C_j^{\mu} + C_j^{\mu-1}$, it follows easily from this that

$$\sigma_{k,j+1,\nu} = (\nu - j)\sigma_{k,j,\nu} - k\sigma_{k-1,j,j-1}.$$

This recursion relation enables us to prove (1.51) easily by induction.

Suppose that $\nu \leqslant j - 1$. Then $k \leqslant \nu < j$, and the required equality (1.50) reduces (unlike (1.51)) to the form

$$\sum_{\mu=0}^{k} (-1)^{\mu} \frac{C_k^{\mu}}{(j - \mu)(j - \mu - 1) \cdots (\nu - \mu + 1)}$$
$$= \frac{(\nu - k)!}{j!} (\nu - j)(\nu - j - 1) \cdots (\nu - j - k + 1).$$

This can be proved by induction similarly to (1.51).

10. The Fourier transform of the functions $|x|^{\beta} Y_m(x')$. Let $\alpha \in C^1$. We have

$$F\left(\frac{Y_m(x')}{|x|^{n-\alpha}} \right)$$
$$= \begin{cases} A_{n,m}(\alpha) \dfrac{Y_m(x')}{|x|^{\alpha}} & \text{for } \alpha + m \neq 0, -2, -4, -6, \ldots \\ & \text{and } \alpha - m - n \neq 0, 2, 4, 6, \ldots, \\ B_{n,m}(\alpha) \left[C_{n,m}(\alpha) + \ln \dfrac{1}{|x|^2} \right] \dfrac{Y_m(x')}{|x|^{\alpha}} & \text{for } \alpha + m = 0, -2, -4, -6, \ldots, \\ (2\pi)^n Y_m(-iD)(-\Delta)((\alpha - m - n)/2)\delta(x) & \text{for } \alpha - m - n = 0, 2, 4, 6, \ldots, \end{cases} \tag{1.52}$$

where the Fourier transformation is understood in the sense of the distributions S', and the constants $A_{n,m}(\alpha)$, $B_{n,m}(\alpha)$, and $C_{n,m}(\alpha)$ are computed by the formulas

$$A_{n,m}(\alpha) = i^m 2^\alpha \pi^{n/2} \frac{\Gamma((\alpha + m)/2)}{\Gamma((n - \alpha + m)/2)},$$

$$B_{n,m}(\alpha) = \frac{(-1)^k i^m 2^{-m-2k} \pi^{n/2}}{\Gamma(k + m + n/2)}, \qquad k = -\frac{m + \alpha}{2},$$

$$C_{n,m}(\alpha) = \ln 4 + \Gamma'(1) + \frac{\Gamma'(n/2 + k)}{\Gamma(n/2 + k)} + \sum_{\nu=1}^{m+k} \frac{1}{\nu} + \sum_{\nu=1}^{m} \frac{(-1)^\nu}{\nu} \frac{C_{m+k}^\nu}{C_{m+k}^k}.$$

The cases $\alpha = 0$ and $0 < \operatorname{Re} \alpha < n$ are well known ([37], [11], [39], [33]); the case $\alpha - m - n = 0, 2, 4, 6, \ldots$ is obvious, since it reduces to the computation of the Fourier transform of a polynomial. The general formula (1.52) was established in [31].

§2. The symbol of a generalized Riesz potential

A *generalized Riesz potential* is defined to be an integral

$$K_\theta^\alpha \varphi = \int_{R^n} \frac{\theta((x - t)/|x - t|)}{|x - t|^{n-\alpha}} \varphi(t)\, dt, \qquad 0 < \alpha < n. \tag{2.1}$$

Following the terminology adopted in [11] for multidimensional singular ($\alpha = 0$) integrals, we call the function $\theta(x')$, $x' = x/|x|$, the *characteristic* of the potential (2.1).

In §§5 and 6 the operator inverse to (2.1) is constructed (hypersingular construction). It is clear that the question of inverting the potentials (2.1) is connected in some way or another with their symbols

$$\mathcal{K}_\theta^\alpha(x) = \int_{R^n} \frac{\theta(t')}{|t|^{n-\alpha}} e^{ix\cdot t}\, dt. \tag{2.2}$$

The present section contains an investigation of the symbol $\mathcal{K}_\theta^\alpha(x)$ and, in particular, its representation by a surface integral over the sphere (see subsection 4).

1. Preliminary discussion. The integral (2.2) may turn out to be divergent at infinity. The difficulties associated with divergence of the Fourier integral are easily resolved by regarding the Fourier transform of the function $\theta(t')|t|^{\alpha-n}$ in the sense of generalized functions. However, for our purposes—an efficient construction of the inverse operator—it is required that the symbol $\mathcal{K}_\theta^\alpha(x)$ be an ordinary and even a sufficiently smooth function on the sphere Σ_{n-1} (the symbol is homogeneous of degree $-\alpha$). For the construction of the operator inverse to the potential in the form of an HSI it turns out to be essential that the symbol behaves nicely: The property of annihilation of the kernel $\theta(t')|t|^{\alpha-n}$ by the corresponding HSI (see §6.1) is based on the smoothness of the symbol. Therefore, we begin with the question of existence of the integral (2.2) in the ordinary sense.

It is clear that the integral (2.2) defines an ordinary function for $0 < \alpha < n/2$ if $\theta(x')$ is bounded (the L_1- and L_2-theory of the Fourier integral is in effect). We shall see below that the integral (2.2) converges conditionally for $0 < \alpha < (n + 1)/2$ for a sufficiently smooth characteristic $\theta(x')$. A salient reason for this is the fact that it is valid in the simplest case of a Riesz potential ($\theta \equiv 1$): The Fourier transform of the

Riesz kernel $|t|^{\alpha-n}$ is a conditionally convergent integral if and only if $0 < \alpha < (n+1)/2$. Indeed,

$$\lim_{N\to\infty}\int_{|t|<N}|t|^{\alpha-n}e^{ix\cdot t}\,dt = \lim_{N\to\infty}\int_0^N \rho^{\alpha-1}\,d\rho\int_{\Sigma_{n-1}} e^{i\rho x\cdot\sigma}\,d\sigma.$$

Using (1.15) and formula 3.915.5 in [6], it is not hard to get that

$$\int_{\Sigma_{n-1}} e^{i\rho x\cdot\sigma}\,d\sigma = 2\pi^{(n-1)/2}\left(\frac{2}{\rho|x|}\right)^{(n-2)/2} J_{(n-2)/2}(\rho|x|).$$

Therefore,

$$\lim_{N\to\infty}\int_{|t|<N}|t|^{\alpha-n}e^{ix\cdot t}\,dt = \frac{(2\pi)^{n/2}}{|x|^{\alpha}}\lim_{N\to\infty}\int_0^{N|x|}\rho^{\alpha-n/2}J_{(n-2)/2}(\rho)\,d\rho$$

and the convergence of the integral as $N\to\infty$ for $0<\alpha<(n+1)/2$ is seen from the asymptotic behavior of the Bessel function at infinity (cf. also formula 6.561.14 in [6]).

The Calderón-Zygmund formula representing a symbol as an integral over the sphere (see (1.41)) is well known for multidimensional singular integrals. The analogous fact for the potentials (2.1) turned out to be connected with divergent integrals over the sphere with a singularity on a planar section of the sphere. Such f.p.-integrals are introduced and studied in subsection 2. A regularization of them is given in terms of the x'-means $M_\theta(x', y)$ introduced in §1.2. Then a basic theorem on representing the symbol $\mathcal{K}_\theta^\alpha(x)$ by the indicated f.p.-integrals is given (subsection 3). Here the following goals are achieved at the same time (for a sufficiently smooth characteristic $\theta(x')$).

a) It is shown that the Fourier transform (2.2) exists as a conditionally convergent integral for $0<\alpha<(n+1)/2$.

b) The representation (2.34)–(2.34′) (obtained for $0<\alpha<(n+1)/2$) of the symbol $\mathcal{K}_\theta^\alpha(x)$ by surface f.p.-integrals which make sense for $\alpha \geqslant (n+1)/2$ provides an analytic continuation of $\mathcal{K}_\theta^\alpha(x)$ with respect to α into the strip $0<\operatorname{Re}\alpha<\lambda$, where λ is the exponent of smoothness of the characteristic $\theta(t')$. For $\alpha\geqslant(n+1)/2$ the symbol $\mathcal{K}_\theta^\alpha(x)$ is understood as the function (2.34)–(2.34′) everywhere below.

c) It is shown that smoothness of the symbol is ensured by sufficient smoothness of the characteristic (see (2.86)).

Moreover, in subsection 4 we take care that the function $\mathcal{K}_\theta^\alpha(x)$ be the Fourier transform of the kernel $\theta(t')|t|^{\alpha-n}$ in the generalized sense for all admissible $0<\alpha<n$.

2. The Hadamard constructions of divergent integrals over the sphere. We consider the integrals

$$J(x) = \int_{\Sigma_{n-1}}\frac{\theta(\sigma)}{(-ix\cdot\sigma)^{\alpha}}\,d\sigma, \qquad \alpha>0, \tag{2.3}$$

where $\theta(\sigma)$ is a function defined on Σ_{n-1}, $x\in R^n$, and $(-iy)^\alpha$ is understood as in (1.17). For $\alpha\geqslant 1$ the integral in (2.3) is divergent: the integrand has a singularity of order α on the $(n-2)$-dimensional section of Σ_{n-1} by the hyperplane $\sigma\cdot x=0$ orthogonal to the vector x.

To interpret the integral (2.3) for $\alpha \geqslant 1$ in the spirit of the Cauchy-Hadamard construction, we excise from Σ_{n-1} a "hoop" containing the section $\sigma \cdot x = 0$, $|\sigma| = 1$, by means of the two hyperplanes $\sigma \cdot x = \pm\varepsilon$, and consider the integral

$$J_\varepsilon(x) = \int_{|x\cdot\sigma|>\varepsilon} (-ix\cdot\sigma)^{-\alpha}\theta(\sigma)\,d\sigma. \tag{2.4}$$

Following an idea of Hadamard ([36], §5.5.5), we make the following definition:
DEFINITION 5. If the integral $J_\varepsilon(x)$ admits a representation

$$J_\varepsilon(x) = A_\varepsilon(x) + \sum_{k=1}^{N} a_k(x)\varepsilon^{-\lambda_k} + b(x)\ln\frac{1}{\varepsilon},$$

for which $\operatorname{Re}\lambda_k > 0$ and $A_0(x) = \lim_{\varepsilon\to 0} A_\varepsilon(x)$ exists, then we say that the *finite part* (f.p.) of the integral (2.3) exists, and define

$$\text{f.p.}\int_{\Sigma_{n-1}} \frac{\theta(\sigma)\,d\sigma}{(-ix\cdot\sigma)^\alpha} = A_0(x). \tag{2.5}$$

THEOREM 2. *Suppose that $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda > \max(0, \alpha - 1)$. Then the f.p.-integral (2.3) exists in the sense of Definition 5 and has the following representation (regularization):*

$$f.p.\int_{\Sigma_{n-1}} \frac{\theta(\sigma)\,d\sigma}{(-ix\cdot\sigma)^\alpha} = \int_{\Sigma_{n-1}} \frac{\theta(\sigma) - \sum_{k=0}^{[\alpha]-1}(x'\cdot\sigma)^k/k!\cdot\partial^k M_\theta(x',0)/\partial y^k}{(-ix\cdot\sigma)^\alpha}\,d\sigma$$
$$+\frac{2\pi^{(n+1)/2}}{|x|^\alpha}\sum_{k=0}^{[\alpha]-1}{}' \frac{i^k}{k!\Gamma((\alpha+1-k)/2)\Gamma((n-\alpha+k)/2)}$$
$$\cdot\frac{\partial^k M_\theta(x',0)}{\partial y^k}, \tag{2.6}$$

where the prime on the summation sign means that the terms with indices $k = \alpha - 1$, $\alpha - 3, \ldots$ are omitted in the case of integral α.

PROOF. It is necessary to consider the case $\alpha \geqslant 1$. Applying Lemma 4 to the integral (2.4), we have

$$\int_{|x\cdot\sigma|>\varepsilon} \frac{\theta(\sigma)\,d\sigma}{(-ix\cdot\sigma)^\alpha} = \frac{|\Sigma_{n-2}|}{|x|^\alpha}\int_{\varepsilon/|x|<|y|<1} (1-y^2)^{(n-3)/2}\frac{M_\theta(x',y)}{(-iy)^\alpha}\,dy. \tag{2.7}$$

From this,

$$\int_{|x\cdot\sigma|>\varepsilon} \frac{\theta(\sigma)\,d\sigma}{(-ix\cdot\sigma)^\alpha} = \frac{|\Sigma_{n-2}|}{|x|^\alpha}$$
$$\times\int_{\varepsilon/|x|<|y|<1} (1-y^2)^{(n-3)/2}\frac{M_\theta(x',y) - \sum_{k=0}^{[\alpha]-1}(y^k/k!)\partial^k M_\theta(x',0)/\partial y^k}{(-iy)^\alpha}\,dy$$
$$+\frac{|\Sigma_{n-2}|}{|x|^\alpha}\sum_{k=0}^{[\alpha]-1}\frac{1}{k!}\frac{\partial^k M_\theta(x',0)}{\partial y^k}\int_{\varepsilon/|x|<|y|<1}\frac{(1-y^2)^{(n-3)/2}y^k}{(-iy)^\alpha}\,dy. \tag{2.8}$$

The integral in the second term is equal to zero for integral α and $k = \alpha - 1, \alpha - 3, \ldots$. But if α is not an integer or α is an integer but $k \neq \alpha - 1, \alpha - 3, \ldots$, then simple computations with formulas 8.391 and 9.131.2 in [6] taken into account give us that

$$\int_{\varepsilon<|y|<1} \frac{(1-y^2)^{(n-3)/2} y^k}{(-iy)^\alpha} dy = i^k \cos\frac{k-\alpha}{2}\pi \frac{\Gamma((n-1)/2)\Gamma((k-\alpha+1)/2)}{\Gamma((n-\alpha+k)/2)}$$
$$+ \frac{2i^k \cos(k-\alpha)/2(\pi)}{\alpha-k-1} \frac{(1-\varepsilon^2)^{(n-1)/2}}{\varepsilon^{\alpha-1-k}} F\left(1, \frac{n-\alpha+k}{2}; \frac{k-\alpha+3}{2}; \varepsilon^2\right).$$

Then it is not hard to conclude that

$$\text{f.p.} \int_{-1}^{1} \frac{(1-y^2)^{(n-3)/2} y^k}{(-iy)^\alpha} dy$$
$$= \begin{cases} \dfrac{\pi\Gamma((n-1)/2)i^k}{\Gamma((\alpha+1-k)/2)\Gamma((n-\alpha+k)/2)} & \text{if } \alpha \text{ is not an integer, or if } \alpha \text{ is an integer but } k \neq \alpha-1, \alpha-3, \ldots, \\ 0 \quad \text{if } \alpha \text{ is an integer and } k = \alpha-1, \alpha-3, \ldots. \end{cases} \tag{2.9}$$

Therefore, the second line of (2.8) gives rise as $\varepsilon \to 0$ to the second line of the required representation (2.6). We show that the first lines also coincide (after taking the limit as $\varepsilon \to 0$). To do this it suffices, by (2.7), to verify that

$$\int_{|x\cdot\sigma|>\varepsilon} \frac{(x'\cdot\sigma)^k}{(-ix'\cdot\sigma)^\alpha} d\sigma = |\Sigma_{n-2}| \int_{\varepsilon/|x|<|y|<1} \frac{(1-y^2)^{(n-3)/2} y^k}{(-iy)^\alpha} dy,$$

which follows from (1.15). It remains to see that the first line in (2.8) converges as $\varepsilon \to 0$. This follows from Lemma 7, and Theorem 2 is proved.

The regularization obtained in Theorem 2 for the f.p.-integral is realized in terms of an integration over the sphere. It follows from (2.7)–(2.9) that this regularization (2.6) can be written in terms of a one-dimensional integration:

$$\text{f.p.} \int_{\Sigma_{n-1}} \frac{\theta(\sigma)\, d\sigma}{(-ix'\cdot\sigma)^\alpha}$$
$$= |\Sigma_{n-2}| \int_{-1}^{1} (1-y^2)^{(n-3)/2} \frac{M_\theta(x', y) - \Sigma_{k=0}^{[\alpha]-1}(y^k/k!)\partial^k M_\theta(x', 0)/\partial y^k}{(-iy)^\alpha} dy$$
$$+ 2\pi^{(n+1)/2} \sum_{k=0}^{[\alpha]-1}{}' \frac{i^k}{k!\Gamma((\alpha+k-1)/2)\Gamma((n-\alpha+k)/2)} \frac{\partial^k M_\theta(x', 0)}{\partial y^k}, \tag{2.10}$$

where the prime on the summation sign means, as before, that the terms for $k = \alpha - 1, \alpha - 3, \ldots$ are omitted in the case of integral α.

In subsection 3 we shall also make essential use of another regularization:

$$\text{f.p.} \int_{\Sigma_{n-1}} \frac{\theta(\sigma)\, d\sigma}{(-ix'\cdot\sigma)^\alpha} = \int_{-1}^{1} \frac{\tilde{M}_\theta(x', y) - \Sigma_{k=0}^{[\alpha]-1}(y^k/k!)\partial^k \tilde{M}_\theta(x', 0)/\partial y^k}{(-iy)^\alpha} dy$$
$$+ 2 \sum_{k=0}^{[\alpha]-1}{}' \frac{i^k \cos((\alpha-k)/2)\pi}{k!(k-\alpha+1)} \frac{\partial^k \tilde{M}_\theta(x', 0)}{\partial y^k}, \tag{2.11}$$

where the prime on the summation sign means that the term with $k = \alpha - 1$ is omitted in the case of integral α, and

$$\tilde{M}_\theta(x', y) = |\Sigma_{n-2}|(1 - y^2)^{(n-3)/2} M_\theta(x', y). \tag{2.12}$$

The representation (2.11) is obtained from (2.7) by a procedure completely analogous to that in (2.8). In certain problems it turns out to be more convenient to use the functions $\tilde{M}_\theta(x', y)$ instead of $M_\theta(x', y)$ (see (2.15) and (2.34′)). We mention the useful formula

$$\frac{\partial^k \tilde{M}_\theta(x',0)}{\partial y^k} = 2\pi^{(n-1)/2} \sum_{j=0}^{[k/2]} \frac{k!}{j!} \frac{(-1)^j}{\Gamma(k-2j+1)\Gamma((n-1)/2 - j)} \frac{\partial^{k-2j} M_\theta(x',0)}{\partial y^{k-2j}}, \tag{2.13}$$

which expresses $\partial^k \tilde{M}_\theta(x',0)/\partial y^k$ in terms of $\partial^j M_\theta(x',0)/\partial y^j$ and can be obtained with the help of the Leibniz formula.

Remark 7. The regularization (2.6) contains a conditionally convergent integral (which is not absolutely convergent). Thus, taking $1 < \alpha < 2$ for simplicity, we have

$$\int_{\Sigma_{n-1}} \frac{|\theta(\sigma) - M_\theta(x',0)|}{|x' \cdot \sigma|^\alpha} d\sigma \equiv \infty,$$

for example, for $\theta(\sigma) = \sigma_j$. As for the "one-dimensional" regularizations (2.10) and (2.11), they contain absolutely convergent integrals.

Remark 8. It is possible to regard α as complex in (2.3), with $(-iy)^\alpha$ understood as in (1.17). The preceding considerations, in particular, Theorem 2, remain in force for $\lambda > \max(0, \operatorname{Re}\alpha - 1)$.

Remark 9. Let us compare the generalized function f.p. $1/(-iy)^\alpha$ with the following more common distributions in the theory of generalized functions:

$$\text{f.p.}\frac{1}{|y|^\alpha}, \qquad \text{f.p.}\frac{\operatorname{sgn} y}{|y|^\alpha}, \qquad \text{f.p.}\frac{1}{y_\pm^\alpha}. \tag{2.14}$$

Here it is convenient for us to regard them on the test space $C_0^\infty(-1, 1)$. It is easy to compute

$$\text{f.p.}\int_{-1}^{1} \frac{\varphi(y)}{(-iy)^\alpha} dy = \int_{-1}^{1} \frac{\varphi(y) - \sum_{k=0}^{[\alpha]-1} (\varphi^{(k)}(0)/k!) y^k}{(-iy)^\alpha} dy + 2 \sum_{k=0}^{[\alpha]-1}{}' \frac{i^k \cos((k-\alpha)/2)\pi}{k!(k-\alpha+1)} \varphi^{(k)}(0),$$

where the prime on the summation sign means that the term with $k = \alpha - 1$ is omitted in the case of integral α (cf. (2.11)!). The f.p.-integrals for (2.14) are written similarly. The generalized functions (2.14), which are analytic in α everywhere except at $\alpha = 1, 3, 5, \ldots$, $\alpha = 2, 4, 6, \ldots$, and $\alpha = 1, 2, 3, \ldots$, respectively, have poles at the excluded points. However, the generalized function $1/(-iy)^\alpha$ used by us has not poles but removable discontinuities at integral α. Namely, it is not hard to derive from the representation written above that

$$\lim_{\alpha\to m} \text{f.p.}\frac{1}{(-iy)^\alpha} = \frac{1}{(-i)^m}\text{f.p.}\frac{1}{y^m} + \frac{\pi i^{m-1}}{(m-1)!}\delta^{(m-1)}.$$

The situation is analogous with the spherical f.p.-constructions (2.3). Namely,

COROLLARY TO THEOREM 2. *The* f.p.*-integral* (2.3) *with a function* $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda > 0$, *is an analytic function of the parameter* α *in the half-plane* $\operatorname{Re}\alpha < 1 + \lambda$ *with the exception of the values* $\alpha = 1, 2, 3, \ldots$, *at which it has removable discontinuities*:

$$\lim_{\alpha\to m} \text{f.p.}\int_{\Sigma_{n-1}} \frac{\theta(\sigma)\,d\sigma}{(-ix'\cdot\sigma)^\alpha} = \text{f.p.}\int_{\Sigma_{n-1}} \frac{\theta(\sigma)\,d\sigma}{(-ix'\cdot\sigma)^m} + \frac{\pi i^{m-1}}{(m-1)!}\frac{\partial^{m-1}\tilde{M}_\theta(x',0)}{\partial y^{m-1}}, \tag{2.15}$$

where $\tilde{M}_\theta(x', y)$ *is the function in* (2.12).

PROOF. Let us add $\sum_{k=[\operatorname{Re}\alpha]}^{N-1}$ to and subtract it from the sum $\sum_{k=0}^{[\operatorname{Re}\alpha]-1}$ under the integral sign in (2.11), taking $[\operatorname{Re}\alpha] \leq N \leq \lambda + 1$ (so as not to diminish the smoothness of $\theta(\sigma)$). If we then compute the terms subtracted, we get that (2.11) holds with $[\alpha]$ replaced by N for $[\operatorname{Re}\alpha] \leq N \leq \lambda + 1$. Since $[\operatorname{Re}\alpha] \leq [\lambda] + 1$, it is possible to choose $N = [\lambda] + 1$ independently of α. Then in the formula (2.11) so transformed the integral will obviously be analytic in α, and the sum $\sum'^{[\lambda]}_{k=0}$ outside the integral will have removable discontinuities at $\alpha = 1, 2, 3, \ldots$ because the term with index $k = \alpha - 1$ is omitted in the case of integral α. This gives (2.15).

The integral (2.3) can be differentiated by the formula

$$\frac{\partial}{\partial x_k}\text{f.p.}\int_{\Sigma_{n-1}} \frac{\theta(\sigma)\,d\sigma}{(-ix\cdot\sigma)^\alpha} = \alpha i\,\text{f.p.}\int_{\Sigma_{n-1}} \frac{\sigma_k\theta(\sigma)}{(-ix\cdot\sigma)^{\alpha+1}}\,d\sigma. \tag{2.16}$$

We show that it holds for $\alpha < 1$ when $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda > \alpha$ (though it is apparently true for any α). The formula is obvious in the case $\alpha < 0$ (and trivial for $\alpha = 0$). Therefore, its validity for $0 < \alpha < \min(1, \lambda)$ will be ensured if we can show that the left-hand and right-hand sides are analytic in α for $\operatorname{Re}\alpha < \min(1, \lambda)$. The analyticity of the right-hand side was noted in the corollary to Theorem 2, and that of the left-hand side can be seen after applying (1.35).

Let us regard an f.p.-integral as an operator acting on the functions $\theta(\sigma)$ defined on the sphere. The next theorem illustrates the loss of smoothness of a function $\theta(\sigma)$ when this operator is applied to it.

THEOREM 3. *Suppose that* $\alpha > 0$ *and* $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda > \max(0, \alpha - 1)$. *Then*

$$f.p.\int_{\Sigma_{n-1}} \frac{\theta(\sigma)\,d\sigma}{(-ix'\cdot\sigma)^\alpha} \in C^{\lambda+1-\alpha}(\Sigma_{n-1}) \tag{2.17}$$

under the conditions that $\alpha \neq 1, 2, 3, \ldots$ *and* $\lambda - \alpha \neq 0, 1, 2, \ldots$. *But if* α *or* $\lambda - \alpha$ *is an integer, then* $C^{\lambda+1-\alpha}(\Sigma_{n-1})$ *should be replaced by* $C_*^{\lambda+1-\alpha}(\Sigma_{n-1})$ *in* (2.17). *It is possible to take* $\lambda = 0$ *in the case* $0 < \alpha < 1$.

PROOF. By Lemma 2, it suffices for us to consider only the first term in the representation (2.6) of the integral (2.17); moreover, it suffices to consider the integral

$$J(x)=\int_{-1/2}^{1/2}\frac{M_\theta(x',y)-\sum_{k=0}^{[\alpha]-1}(y^k/k!)\partial^k M_\theta(x',0)/\partial y^k}{(-iy)^\alpha}dy. \tag{2.18}$$

Let us integrate by parts $[\alpha]-1$ times here (assuming that $\alpha\geqslant 2$):

$$J(x)=\sum_{k=0}^{[\alpha]-2}\left[a_k\frac{\partial^k M_\theta(x',1/2)}{\partial y}+b_k\frac{\partial^k M_\theta(x',-1/2)}{\partial y}\right]+cJ_1(x), \tag{2.19}$$

where a_k, b_k, and c are constants, and

$$J_1(x)=\int_{-1/2}^{1/2}\left[\frac{\partial^{[\alpha]-1}M_\theta(x',y)}{\partial y^{[\alpha]-1}}-\frac{\partial^{[\alpha]-1}M_\theta(x',0)}{\partial y^{[\alpha]-1}}\right]\frac{dy}{(-iy)^{\alpha-[\alpha]+1}} \tag{2.20}$$

(but if $0<\alpha<2$, we leave the integral (2.18) unchanged: $J=J_1$). The terms outside the integral in (2.19) belong to $C^{\lambda-\alpha+1}(\Sigma_{n-1})$, by Lemma 3. It remains to consider the integral $J_1(x)$.

I. *The case* $\alpha\neq 1,2,3,\ldots$ *and* $\lambda-\alpha\neq 0,1,2,\ldots$. It must be proved that

$$D^m J_1(x)\in C^{\mu-[\mu]}(\Sigma_{n-1}) \tag{2.21}$$

for all multi-indices m of length $|m|=[\mu]$, where $\mu=\lambda-\alpha+1$. We have

$$D^m J_1(x)=\int_{-1/2}^{1/2}\frac{f(x,y)-f(x,0)}{(-iy)^{1+\{\alpha\}}}dy,\qquad \{\alpha\}=\alpha-[\alpha], \tag{2.22}$$

where

$$f(x,y)=D_x^m(\partial/\partial y)^{[\alpha]-1}M_\theta(x',y).$$

In the case $0<\alpha<1$ the derivative $D^mJ_1(x)$ does not have the form (2.22). This case will be handled separately.

The subsequent arguments differ for $\{\lambda\}>\{\alpha\}$ and $\{\lambda\}<\{\alpha\}$. We begin with the simpler former case.

1°. *The case* $\{\lambda\}>\{\alpha\}$ *for* $\alpha>1$. By Lemma 3,

$$f(x,y)\in C^{\lambda-[\mu]-[\alpha]+1}\left(\Sigma_{n-1}\times\left[-\tfrac{1}{2},\tfrac{1}{2}\right]\right).$$

Obviously, $\lambda-[\mu]-[\alpha]+1=\lambda-[\alpha]-[\lambda-\alpha]>0$. Moreover, in our case $\lambda-[\mu]-[\alpha]+1=\{\lambda\}<1$. Checking the required Hölder property for (2.20), we have

$$\begin{aligned}|D^mJ_1(x)-D^mJ_1(z)|\leqslant&\int_{|y|<|x'-z'|}\frac{|f(x',y)-f(x',0)|}{|y|^{1+\{\alpha\}}}dy\\&+\int_{|y|<|x'-z'|}\frac{|f(z',y)-f(z',0)|}{|y|^{1+\{\alpha\}}}dy\\&+\int_{|x'-z'|<|y|<1/2}\frac{|f(x',y)-f(z',y)|}{|y|^{1+\{\alpha\}}}dy\\&+\int_{|x'-z'|<|y|<1/2}\frac{|f(x',0)-f(z',0)|}{|y|^{1+\{\alpha\}}}dy\\&\leqslant 2c\int_0^{|x'-z'|}\frac{y^{\{\lambda\}}dy}{y^{1+\{\alpha\}}}+2c|x'-z'|^{\{\lambda\}}\int_{|x'-z'|}^{1/2}y^{-1-\{\alpha\}}dy\\&\leqslant c_1|x'-z'|^{\{\lambda\}-\{\alpha\}},\end{aligned} \tag{2.23}$$

which yields (2.21), since $\mu - [\mu] = \{\lambda\} - \{\alpha\}$ in this case.

2°. *The case* $\{\lambda\} > \{\alpha\}$ *for* $0 < \alpha < 1$. In this case we differentiate the f.p.-integral once at the very start according to formula (2.16), i.e., we consider the function

$$\frac{\partial}{\partial x_k}\int_{\Sigma_{n-1}}\frac{\theta(\sigma)}{(-ix\cdot\sigma)^{\alpha}}d\sigma, \qquad k = 1,2,\ldots,n$$

(note that $\lambda - \alpha + 1 > 1$ for $\{\lambda\} > \{\alpha\}$). By (2.16), this leads to the consideration of the integral

$$J(x) = \int_{-1/2}^{1/2}\frac{M_{\theta_k}(x',y) - M_{\theta_k}(x',0)}{(-iy)^{1+\alpha}}dy$$

instead of (2.18), where $\theta_k(\sigma) = \sigma_k\theta(\sigma) \in C^{\lambda}(\Sigma_{n-1})$.

For this integral the condition (2.21) must now be checked for all multi-indices m of length 1 less than $[\lambda - \alpha + 1]$, i.e., $|m| = [\lambda - \alpha]$. This is done analogously to the estimates in (2.23): The corresponding function $f(x, y) = D_x^m M_{\theta_k}(x', y)$ now belongs to the class $C^{\lambda-[\mu]+1}(\Sigma_{n-1}\times[-\frac{1}{2},\frac{1}{2}])$.

3°. *The case* $\{\lambda\} < \{\alpha\}$ *for* $0 < \alpha < 1$. Let

$$g(x') = \int_{\Sigma_{n-1}}\frac{\theta(\sigma)\,d\sigma}{(-ix'\cdot\sigma)^{\alpha}} \quad\text{and}\quad x_h' = \frac{x+h}{|x+h|}, \qquad h \in R^n.$$

It must be shown that

$$|g(x') - g(x_h')| \leqslant c|h|^{\lambda+1-\alpha} \tag{2.24}$$

for all $x' \in \Sigma_{n-1}$ and $h \in R^n$ ($|h| \leqslant 1$), where c does not depend on x' or h. We have

$$g(x') - g(x_h') = \int_{\Sigma_{n-1}}\theta(\sigma)\left[\frac{1}{(-ix'\cdot\sigma)^{\alpha}} - \frac{1}{(-ix_h'\cdot\sigma)^{\alpha}}\right]d\sigma.$$

The same operations used for deriving (1.35) are carried out in this integral. Applying the rotation $\sigma = \mathrm{rot}_x\,\sigma_{\mathrm{new}}$ for this, we get

$$g(x') - g(x_h') = \int_{\Sigma_{n-1}}\theta_x(\sigma)\left[\frac{1}{(-i\sigma_1)^{\alpha}} - \frac{|x+h|^{\alpha}}{(-i(\sigma_1 + h\,\mathrm{rot}_x\,\sigma))^{\alpha}}\right]d\sigma, \tag{2.25}$$

since $x_h'\cdot\sigma = (x\cdot\sigma + h\cdot\sigma)|x+h|^{-1}$, and we can assume that $|x| = 1$ (see (1.28)). Since

$$||x+h|^{\alpha} - 1| = |(1 + 2x\cdot h + |h|^2)^{\alpha/2} - 1| \leqslant c|h|$$

for $|x| = 1$, we can replace $|x+h|^{\alpha}$ by 1 in (2.25). In the integral thus obtained, denoted by J, we pass to integration over the hemispheres $\sigma_n > 0$ and $\sigma_n < 0$. If we then project onto the $(n-1)$-dimensional ball lying in their base, we can pass to iterated integration (over σ_1 and with respect to the collection of remaining variables) in the resulting $(n-1)$-fold integral.(1) The result is

(1) The subsequent expressions relate to the case $n \geqslant 3$; the planar case $n = 2$ is simpler and can be treated similarly with the appropriate simplifications.

$$J(x)=\int_{-1/2}^{1/2}\frac{M_\theta(x',y)-\sum_{k=0}^{[\alpha]-1}(y^k/k!)\partial^k M_\theta(x',0)/\partial y^k}{(-iy)^\alpha}dy. \quad (2.18)$$

Let us integrate by parts $[\alpha]-1$ times here (assuming that $\alpha\geqslant 2$):

$$J(x)=\sum_{k=0}^{[\alpha]-2}\left[a_k\frac{\partial^k M_\theta(x',1/2)}{\partial y}+b_k\frac{\partial^k M_\theta(x',-1/2)}{\partial y}\right]+cJ_1(x), \quad (2.19)$$

where a_k, b_k, and c are constants, and

$$J_1(x)=\int_{-1/2}^{1/2}\left[\frac{\partial^{[\alpha]-1}M_\theta(x',y)}{\partial y^{[\alpha]-1}}-\frac{\partial^{[\alpha]-1}M_\theta(x',0)}{\partial y^{[\alpha]-1}}\right]\frac{dy}{(-iy)^{\alpha-[\alpha]+1}} \quad (2.20)$$

(but if $0<\alpha<2$, we leave the integral (2.18) unchanged: $J=J_1$). The terms outside the integral in (2.19) belong to $C^{\lambda-\alpha+1}(\Sigma_{n-1})$, by Lemma 3. It remains to consider the integral $J_1(x)$.

I. *The case* $\alpha\neq 1,2,3,\dots$ *and* $\lambda-\alpha\neq 0,1,2,\dots$. It must be proved that

$$D^m J_1(x)\in C^{\mu-[\mu]}(\Sigma_{n-1}) \quad (2.21)$$

for all multi-indices m of length $|m|=[\mu]$, where $\mu=\lambda-\alpha+1$. We have

$$D^m J_1(x)=\int_{-1/2}^{1/2}\frac{f(x,y)-f(x,0)}{(-iy)^{1+\{\alpha\}}}dy, \qquad \{\alpha\}=\alpha-[\alpha], \quad (2.22)$$

where

$$f(x,y)=D_x^m(\partial/\partial y)^{[\alpha]-1}M_\theta(x',y).$$

In the case $0<\alpha<1$ the derivative $D^m J_1(x)$ does not have the form (2.22). This case will be handled separately.

The subsequent arguments differ for $\{\lambda\}>\{\alpha\}$ and $\{\lambda\}<\{\alpha\}$. We begin with the simpler former case.

1°. *The case* $\{\lambda\}>\{\alpha\}$ *for* $\alpha>1$. By Lemma 3,

$$f(x,y)\in C^{\lambda-[\mu]-[\alpha]+1}\left(\Sigma_{n-1}\times\left[-\tfrac12,\tfrac12\right]\right).$$

Obviously, $\lambda-[\mu]-[\alpha]+1=\lambda-[\alpha]-[\lambda-\alpha]>0$. Moreover, in our case $\lambda-[\mu]-[\alpha]+1=\{\lambda\}<1$. Checking the required Hölder property for (2.20), we have

$$\begin{aligned}|D^mJ_1(x)-D^mJ_1(z)|&\leqslant\int_{|y|<|x'-z'|}\frac{|f(x',y)-f(x',0)|}{|y|^{1+\{\alpha\}}}dy\\&+\int_{|y|<|x'-z'|}\frac{|f(z',y)-f(z',0)|}{|y|^{1+\{\alpha\}}}dy\\&+\int_{|x'-z'|<|y|<1/2}\frac{|f(x',y)-f(z',y)|}{|y|^{1+\{\alpha\}}}dy\\&+\int_{|x'-z'|<|y|<1/2}\frac{|f(x',0)-f(z',0)|}{|y|^{1+\{\alpha\}}}dy\\&\leqslant 2c\int_0^{|x'-z'|}\frac{y^{\{\lambda\}}dy}{y^{1+\{\alpha\}}}+2c|x'-z'|^{\{\lambda\}}\int_{|x'-z'|}^{1/2}y^{-1-\{\alpha\}}dy\\&\leqslant c_1|x'-z'|^{\{\lambda\}-\{\alpha\}},\end{aligned} \quad (2.23)$$

$$J=\int_{-1}^{1}dy\int_{|\xi|<\sqrt{1-y^2}}\theta_x(\sigma_+)\left\{\frac{1}{(-iy)^\alpha}-\frac{1}{[-i(y+h\,\mathrm{rot}_x\,\sigma_+)]^\alpha}\right\}\frac{d\xi}{\sqrt{1-y^2-|\xi|^2}}$$

$$+\int_{-1}^{1}dy\int_{|\xi|<\sqrt{1-y^2}}\theta_x(\sigma_-)\left\{\frac{1}{(-iy)^\alpha}-\frac{1}{[-i(y+h\,\mathrm{rot}_x\,\sigma_-)]^\alpha}\right\}\frac{d\xi}{\sqrt{1-y^2-|\xi|^2}},$$

where $\sigma_\pm=(y,\xi,\pm\sqrt{1-y^2-|\xi|^2})$ and $\xi=(\xi_1,\ldots,\xi_{n-2})$.

Performing the substitution $\xi\to\xi\sqrt{1-y^2}$ and then changing the order of integration, we get $J=J_+ + J_-$, where

$$J_\pm=\int_{B_{n-2}(0,1)}\frac{d\xi}{\sqrt{1-|\xi|^2}}$$

$$\times\int_{-1}^{1}(1-y^2)^{(n-3)/2}\left\{\frac{1}{(-iy)^\alpha}-\frac{1}{[-i(y+h\,\mathrm{rot}_x\,z_\pm)]^\alpha}\right\}\theta_x(z_\pm)\,dy,$$

and $z_\pm$ are the points in (1.29). The integrals J_+ and J_- are of a single type, so we estimate only J_+. Let $\mathcal{K}$ denote the inside integral in J_+, so that $\mathcal{K}=\mathcal{K}(x,\xi,h)$. It suffices to prove that $|\mathcal{K}(x,\xi,h)|\leq c|h|^{\lambda-\alpha+1}$, where c does not depend on x, ξ, or h. We have

$$\mathcal{K}=\int_{-1}^{1}\left[(1-y^2)^{(n-3)/2}-1\right]\theta_x(z_+)\left\{\frac{1}{(-iy)^\alpha}-\frac{1}{[-i(y+h\,\mathrm{rot}_x\,z_+)]^\alpha}\right\}dy$$

$$+\int_{-1}^{1}\left[\theta_x(z_+)-\theta_x(z_+)|_{y=0}\right]\left\{\frac{1}{(-iy)^\alpha}-\frac{1}{[-i(y+h\,\mathrm{rot}_x\,z_+)]^\alpha}\right\}dy$$

$$+\theta_x(z_+)|_{y=0}\int_{-1}^{1}\left\{\frac{1}{(-iy)^\alpha}-\frac{1}{[-i(y+h\,\mathrm{rot}_x\,z_+)]^\alpha}\right\}dy$$

$$\stackrel{\text{def}}{=}\mathcal{K}_1+\mathcal{K}_2+\theta_x(z_+)|_{y=0}\mathcal{K}_3.$$

Estimate of $\mathcal{K}_1$. Since

$$|(1-y^2)^{(n-3)/2}-1|\leq c|y|$$

($|(1-y^2)^{-1/2}-1|\leq c|y|(1-y^2)^{-1/2}$ for $n=2$), the boundedness of the function $\theta_x(z_+)$ gives us that

$$|\mathcal{K}_1|\leq c\int_{-1}^{1}|y|\left|\frac{1}{(-iy)^\alpha}-\frac{1}{[-i(y+h\,\mathrm{rot}_x\,z_+)]^\alpha}\right|dy. \tag{2.26}$$

From this, after the substitution $y=|h|\tau$, we find that

$$|\mathcal{K}_1|\leq c|h|^{2-\alpha}\int_{-1/|h|}^{1/|h|}|\tau|\left|\frac{1}{(-i\tau)^\alpha}-\frac{1}{[-i(\tau+h'\,\mathrm{rot}_x\,\tilde{z}_+)]^\alpha}\right|d\tau,$$

where $h'=h/|h|$ and

$$\tilde{z}_+=\left(|h|\tau,\xi\sqrt{1-|h|^2\tau^2},\sqrt{1-|h|^2\tau^2}\sqrt{1-|\xi|^2}\right).$$

Obviously,

$$|h'\,\mathrm{rot}_x\,\tilde{z}_+|\leq 1. \tag{2.27}$$

Further,

$$\left|\frac{1}{(-ia)^\alpha}-\frac{1}{(-ib)^\alpha}\right|\leqslant\left|\frac{1}{|a|^\alpha}-\frac{1}{|b|^\alpha}\right|+\frac{1}{|a|^\alpha}\left|e^{(\alpha\pi/2)\,\mathrm{sgn}\,a}-e^{(\alpha\pi/2)\,\mathrm{sgn}\,b}\right|;$$

the second term here disappears if a and b have the same sign. In our case $a=\tau$, $b=\tau+h'\cdot\mathrm{rot}_x\tilde{z}_+$, and for $|\tau|$ sufficiently large ($|\tau|>2$, by (2.27)) the signs of τ and $\tau+h'\,\mathrm{rot}_x\tilde{z}_+$ coincide. Therefore,

$$|\mathcal{K}_1|\leqslant c|h|^{2-\alpha}\int_{-1/|h|}^{1/|h|}|\tau|\left|\frac{1}{|\tau|^\alpha}-\frac{1}{|\tau+h'\,\mathrm{rot}_x\tilde{z}_+|^\alpha}\right|d\tau$$

$$+c|h|^{2-\alpha}\int_{-2}^{2}|\tau|\left|\frac{1}{|\tau|^\alpha}+\frac{1}{|\tau+h'\,\mathrm{rot}_x\tilde{z}_+|^\alpha}\right|d\tau.$$

The second term is majorized by $c|h|^{2-\alpha}$; the integral in the first term can be estimated in the same way for $|\tau|<2$, so that

$$|\mathcal{K}_1|\leqslant c|h|^{2-\alpha}+c|h|^{2-\alpha}\int_2^{1/|h|}\tau\left|\tau^{-\alpha}-(\tau+h'\,\mathrm{rot}_x\tilde{z}_+)^{-\alpha}\right|d\tau$$

$$+c|h|^{2-\alpha}\int_2^{1/|h|}\tau\left|\tau^{-\alpha}-(\tau-h'\,\mathrm{rot}_x\tilde{\tilde{z}}_+)^{-\alpha}\right|d\tau, \tag{2.28}$$

where $\tilde{\tilde{z}}_+=(-|h|\tau,\xi\sqrt{1-|h|^2\tau^2},\sqrt{1-|h|^2\tau^2}\sqrt{1-|\xi|^2})$.

The remaining two integrals are of the same type, so we estimate only one of them. For this, let us apply the mean value theorem to the function $f(s)=(\tau+s)^{-\alpha}$. This gives

$$(\tau+s_1)^{-\alpha}-(\tau+s_2)^{-\alpha}=\alpha(\tau+\theta)^{-1-\alpha}(s_2-s_1),$$

where $s_1<\theta<s_2$. For $s_1=0$ and $s_2=h'\,\mathrm{rot}_x\tilde{z}_+$ we obtain

$$\left|\tau^{-\alpha}-(\tau+h'\,\mathrm{rot}_x\tilde{z}_+)^{-\alpha}\right|\leqslant\alpha\tau^{-1-\alpha},$$

and then the first of the integrals in (2.28) is majorized by the quantity $c|h|$; similarly for the second. Therefore, $|\mathcal{K}_1|\leqslant c|h|$.

Estimate of $\mathcal{K}_2$. Let us use the Hölder property of $\theta(\sigma)$:

$$|\theta_x(z_+)-\theta_x(z_+|_{y=0})|=|\theta(\mathrm{rot}_x z_+)-\theta(\mathrm{rot}_x z_+|_{y=0})|$$
$$\leqslant k|\mathrm{rot}_x z_+-\mathrm{rot}_x z_+|_{y=0}|=k|z_+-z_+|_{y=0}|.$$

It is not hard to see that $|z_+-z_+|_{y=0}|\leqslant\sqrt{2}\,|y|$, so that

$$|\mathcal{K}_2|\leqslant c\int_{-1}^{1}|y|^\lambda\left|(-iy)^{-\alpha}-[-i(y+h\,\mathrm{rot}_x z_+)^{-\alpha}]\right|dy.$$

Next, the integral $\mathcal{K}_2$ can be estimated in exactly the same way as the integral (2.26), and for it the estimate $|\mathcal{K}_2|\leqslant c|h|^{\lambda+1-\alpha}$ is obtained for $\lambda<\alpha$, and $|\mathcal{K}_2|\leqslant c|h|\ln 1/|h|$ for $\lambda=\alpha$.

Estimate of $\mathcal{K}_3$. Let us show that $|\mathcal{K}_3|\leqslant c|h|$. We examine in detail the parentheses $(y+h\,\mathrm{rot}_x z_+)$. Let $a_{jk}(x)$ be the coefficients of the matrix of the rotation rot_x. Assume that the sphere is broken up into finitely many subsets Σ_{n-1}^ν, $\nu=1,2,\dots,N$, according to Theorem 1, so that the rotation has coefficients $a_{jk}(x)$ that are infinitely differentiable on the Σ_{n-1}^ν (incidentally, we need only their boundedness). The projections of the vector $\operatorname{rot}_x z_+$ onto the coordinate axes are

$$(\operatorname{rot}_x z_+)_j = ya_{j1}(x) + \sqrt{1-y^2}\left[\sum_{k=2}^{n-1} a_{jk}(x)\xi_{k-1} + \sqrt{1-|\xi|^2}\,a_{jn}(x)\right],$$

$i = 1,\ldots,n$. Hence,

$$h \cdot \operatorname{rot}_x z_+ = yh \cdot \mathbf{a} + \sqrt{1-y^2}\,h \cdot \mathbf{b},$$

where

$$\mathbf{a} = \mathbf{a}(x) = (a_{11}(x),\ldots,a_{n1}(x)), \tag{2.29}$$

and $\mathbf{b} = \mathbf{b}(x, \xi)$ is the vector with components

$$b_j = \sum_{k=2}^{n-1} a_{jk}\xi_{k-1} + \sqrt{1-|\xi|^2}\,a_{jn}(x). \tag{2.30}$$

Obviously, $|\mathbf{a}| = 1$ (since the a_{jk} are the coefficients of a rotation matrix). Also $|\mathbf{b}| = 1$, since $\mathbf{b} = \operatorname{rot}_x z_+|_{y=0}$. Temporarily let

$$A = h \cdot \mathbf{a}, \qquad B = h \cdot \mathbf{b} \qquad (|A| \leqslant |h|,\ |B| \leqslant |h|). \tag{2.31}$$

After the substitution $y = \cos s$, $0 < s < \pi$, the integral $\mathcal{K}_3$ takes the following form in the notation of (2.29)–(2.31):

$$\mathcal{K}_3 = \int_0^\pi \{(-i\cos s)^{-\alpha} - (-i[(1+A)\cos s + B\sin s])^{-\alpha}\}\sin s\, ds.$$

Since

$$(1+A)\cos s + B\sin s = \sqrt{(1+A)^2 + B^2}\,\cos(s-\varphi), \qquad \varphi = \arctan\frac{B}{1+A},$$

it follows that

$$\begin{aligned}\mathcal{K}_3 &= \left[1 - ((1+A)^2 + B^2)^{-1/2}\right]\int_0^\pi \sin s(-i\cos s)^{-\alpha}\, ds \\ &\quad + ((1+A)^2 + B^2)^{-1/2}\int_0^\pi \{\sin s(-i\cos s)^{-\alpha} - \sin s[-i\cos(s-\varphi)]^{-\alpha}\}\, ds \\ &= \mathcal{K}_3' + \mathcal{K}'''_3.\end{aligned}$$

An estimate for $\mathcal{K}_3'$ is clear: $|\mathcal{K}_3'| \leqslant c(|A| + |B|) \leqslant 2c|h|$, while for $\mathcal{K}_3''$ we have

$$\begin{aligned}|\mathcal{K}_3'''| &\leqslant c\left|\int_0^\pi \sin s(-i\cos s)^{-\alpha}\, ds - \int_{-\varphi}^{\pi-\varphi}\frac{\sin(s+\varphi)}{(-i\cos s)^\alpha}\, ds\right| \\ &\leqslant c\left|\int_0^\pi \frac{\sin s\, ds}{(-i\cos s)^\alpha} - \int_{-\varphi}^{\pi-\varphi}\frac{\sin s\, ds}{(-i\cos s)^\alpha}\right| + c\left|\int_{-\varphi}^{\pi-\varphi}\frac{\sin(s+\varphi) - \sin s}{(-i\cos s)^\alpha}\, ds\right| \\ &\leqslant c\left|\int_{\pi-\varphi}^{\pi}\frac{\sin s}{(-i\cos s)^\alpha}\, ds\right| + c\left|\int_{-\varphi}^{0}\frac{\sin s\, ds}{(-i\cos s)^\alpha}\right| + c|\varphi|.\end{aligned}$$

Obviously, $|\varphi| \leqslant |B/(1+A)| \leqslant c|h|$. Further, taking account of the fact that $|\cos s| \geqslant 1/2$ in neighborhoods of $s = 0$ and $s = \pi$, we get $|\mathcal{K}_3''| \leqslant c|\varphi|^2 + c|\varphi| \leqslant c|h|$, which finishes the estimates.

Gathering our estimates together, we get (2.24) for $0 \leqslant \lambda < \alpha < 1$.

REMARK 10. In the case $0 < \lambda = \alpha < 1$ we get that

$$\int_{\Sigma_{n-1}} \theta(\sigma)(-ix'\cdot\sigma)^{-\alpha}\,d\sigma \in C^1_*(\Sigma_{n-1})$$

(see the estimates for $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$).

4°. *The case* $\{\lambda\} < \{\alpha\}$ *for* $\alpha > 1$. Let us show that this case reduces to the preceding one. Integrating by parts once more in (2.20), we come to the integral

$$J_2(x') = \int_{-1/2}^{1/2} \frac{(\partial/\partial y)^{[\alpha]} M_\theta(x', y)}{(-iy)^{\{\alpha\}}}\,dy$$

(note that $[\alpha] \leqslant \lambda$ for $\{\lambda\} < \{\alpha\}$). By Remark 3, it suffices to study the integral

$$J_3(x') = \int_{-1/2}^{1/2} \frac{M_{\theta_1}(x', y)}{(-iy)^{\{\alpha\}}}\,dy,$$

where $\theta_1(\sigma) \in C^{\lambda-[\alpha]}(\Sigma_{n-1})$. Since

$$(1-y^2)^{(n-3)/2} = 1 - \frac{n-3}{2}y^2 + \cdots,$$

it suffices to study the integral

$$J_4(x') = \int_{-1}^{1} \frac{M_{\theta_1}(x', y)}{(-iy)^{\{\alpha\}}}(1-y^2)^{(n-3)/2}\,dy = \frac{1}{|\Sigma_{n-2}|}\int_{\Sigma_{n-2}} \frac{\theta_1(\sigma)\,d\sigma}{(-ix'\cdot\sigma)^{\{\alpha\}}} \tag{2.32}$$

(see (1.35)) using results of an obvious examination of the integral in (2.32) over $1/2 < |y| < 1$. If $\lambda - [\alpha] < 1$, then $J_4(x') \in C^{\lambda-\alpha+1}(\Sigma_{n-1})$ by virtue of case 3°, since $\{\lambda - [\alpha]\} = \{\lambda\} < \{\alpha\}$. Suppose that $\lambda - [\alpha] \geqslant 1$. Let

$$\lambda - [\alpha] = p + \{\lambda\}, \qquad p = [\lambda - [\alpha]] = [\lambda] - [\alpha].$$

Let us perform p times ($|m| = p$) the differentiation D^m of $J_4(x)$ with respect to x. Generally speaking, this can be done by applying (2.16) p times. However, it is possible to avoid the p-fold application of this formula (which is undesirable for $p > 1$ because we have given its proof only for $0 < \alpha < 1$). Let us proceed as follows. After applying (2.16) once, we arrive at an integral of the form

$$\frac{\partial}{\partial x_k} J_4(x) = \text{f.p.}\int_{\Sigma_{n-1}} \theta_2(\sigma)(-ix\cdot\sigma)^{-1-\{\alpha\}}\,d\sigma, \qquad \theta_2(\sigma) \in C^{\lambda-[\alpha]}(\Sigma_{n-1}).$$

For an analysis of it we need to examine

$$J_5(x') = \int_{-1/2}^{1/2} \frac{M_{\theta_2}(x', y) - M_{\theta_2}(x', 0)}{(-iy)^{1+\{\alpha\}}}\,dy.$$

Integration by parts leads to

$$J_6(x') = \int_{-1/2}^{1/2} \frac{\partial}{\partial y} M_{\theta_2}(x', y)(-iy)^{-\{\alpha\}}\,dy.$$

Application of Remark 3 leads to

$$J_7(x') = \int_{-1/2}^{1/2} M_{\theta_3}(x', y)(-iy)^{-\{\alpha\}}\,dy,$$

where $\theta_3(x') \in C^{\lambda-[\alpha]+1}(\Sigma_{n-1})$, and, consequently, to

$$J_8(x') = \int_{\Sigma_{n-1}} \theta_3(\sigma)(-ix'\cdot\sigma)^{-\{\alpha\}}\,d\sigma.$$

Carrying out a similar procedure $p-1$ more times, we come to the necessity of proving that

$$J_9(x') = \int_{\Sigma_{n-1}} \frac{\theta_*(\sigma)\,d\sigma}{(-ix'\cdot\sigma)^{\{\alpha\}}} \in C^{\lambda-\alpha+1-p}(\Sigma_{n-1}),$$

where $\theta_*(\sigma) \in C^{\lambda-[\alpha]-p}(\Sigma_{n-1})$. But this follows from the case 3°, since $\{\lambda-[\alpha]-p\} = \{\lambda\} < \{\alpha\}$ and $\lambda-\alpha+1-p = \{\lambda\}-\{\alpha\}+1$.

II. *The case* $\alpha = 1,2,3,\ldots,\lambda-\alpha \neq 0,1,2,\ldots$. It is necessary to examine the integral (2.22) for $\{\alpha\}=0$. Operations analogous to those in (2.23) yield

$$|(D^mJ_1)(x')-(D^mJ_1)(z')| \leqslant c\,|x'-z'|\ln\frac{2}{|x'-z'|},$$

i.e., $J_1(x') \in C_*^{\lambda-\alpha+1}(\Sigma_{n-1})$.

III. *The case* $\alpha \neq 1,2,3,\ldots$ *and* $\lambda-\alpha = 0,1,2,\ldots$. We apply successive differentiation with respect to x and integration by parts in the integral $J_4(x')$ just as was done in 4°. Performing this operation p times, where $p=\lambda-\alpha$, we arrive at an integral of the form $J_9(x')$, where $\theta_*(\sigma) \in C^{(\lambda)}(\Sigma_{n-1})$. Since $\lambda-\alpha$ is an integer, $\{\lambda\}=\{\alpha\}$. Then $J_9(x') \in C_*^1(\Sigma_{n-1})$ on the basis of Remark 10, and then together with this we have $J_4(x') \in C_*^{\lambda-\alpha+1}(\Sigma_{n-1})$. But then $J(x) \in C_*^{\lambda-\alpha+1}(\Sigma_{n-1})$, too.

IV. *The case* $\alpha = 1,2,3,\ldots$ *and* $\lambda-\alpha = 0,1,2,\ldots$. Now both α and λ are integers. It is necessary to show that $D^mJ_1(x) \in C_*^1(\Sigma_{n-1})$ for orders with $|m| = \lambda-\alpha$. The function $f(x',y)$ in (2.22) is in the class $C^1(\Sigma_{n-1}\times[-\frac{1}{2},\frac{1}{2}])$ for such orders. Repeating the operations in (2.23), we get

$$|(D^mJ_1)(x')-(D^mJ_1)(z')| \leqslant c\,|x'-z'|\ln\frac{2}{|x'-z'|},$$

this time, which is what was required. Theorem 3 is completely proved.

3. Main representation theorem for the symbol of a potential.

THEOREM 4. *Suppose that* $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda > \max(0,\alpha-1)$. *The limit*

$$\mathcal{K}_\theta^\alpha(x') = \lim_{N\to\infty}\int_{|t|<N}\theta(t')\,|t|^{\alpha-n}e^{ix\cdot t}\,dt \tag{2.33}$$

exists uniformly in x' *for* $0<\alpha<(n+1)/2$, *and the following representations are valid*:

$$\mathcal{K}_\theta^\alpha(x') = \Gamma(\alpha)\,f.p.\int_{\Sigma_{n-1}}\frac{\theta(\sigma)\,d\sigma}{(-ix\cdot\sigma)^\alpha},\qquad \alpha\neq 1,2,3,\ldots, \tag{2.34}$$

$$\mathcal{K}_\theta^\alpha(x) = \Gamma(\alpha)\,f.p.\int_{\Sigma_{n-1}}\frac{\theta(\sigma)\,d\sigma}{(-ix\cdot\sigma)^\alpha} + \frac{\pi i^{\alpha-1}}{|x|^\alpha}\frac{\partial^{\alpha-1}\tilde{M}_\theta(x',0)}{\partial y^{\alpha-1}},\qquad \alpha = 1,2,3,\ldots. \tag{2.34'}$$

In particular (*for* $n \geqslant 3$[2]))

$$\mathcal{K}_\theta^1(x) = i\,p.v. \int_{\Sigma_{n-1}} \frac{\theta(\sigma)}{\sigma \cdot x} d\sigma + \frac{\pi}{|x|} \int_{\Sigma^{x'}_{n-2}(0,1)} \theta(\sigma)\, dS_{n-2}, \tag{2.35}$$

$$\mathcal{K}_\theta^2(x) = -f.p. \int_{\Sigma_{n-1}} \frac{\theta(\sigma)}{(\sigma \cdot x)^2} d\sigma + \frac{\pi i}{|x|^2} \int_{\Sigma^{x'}_{n-2}(0,1)} x' \cdot \operatorname{grad} \theta(\sigma)\, dS_{n-2}. \tag{2.36}$$

PROOF. A representation of the symbol $\mathcal{K}_\theta^\alpha(x)$ in the form (2.34) is not hard to obtain in the case $0 < \alpha < 1$. Indeed,

$$\mathcal{K}_\theta^\alpha(x) = \lim_{N \to \infty} \int_{\Sigma_{n-1}} \theta(\sigma)\, d\sigma \int_0^N e^{i\rho\sigma \cdot x} \rho^{\alpha-1}\, d\rho. \tag{2.37}$$

The inside integral converges as $N \to \infty$ for $0 < \alpha < 1$ and is equal to $\Gamma(\alpha) \times (-ix \cdot \sigma)^{-\alpha}$ (see [6], 3.761.4 and 3.761.9). The possibility of taking the limit under the integral sign in (2.37) is based on the Lebesgue dominated convergence theorem, since (we substitute $\rho\sigma \cdot x = \rho_1$)

$$\left| \int_0^N e^{i\rho\sigma \cdot x} \rho^{\alpha-1}\, d\rho \right| \leqslant c\, |\sigma \cdot x|^{-\alpha}, \tag{2.38}$$

where c does not depend on N.

The case $\alpha \geqslant 1$, unlike the case $0 < \alpha < 1$, involves very nontrivial difficulties. In proceeding to this case we shall be guided by the regularization (2.11) of the f.p.-integral. We have the representation

$$\int_{|t|<N} e^{ix \cdot t} |t|^{\alpha-n} \theta(t')\, dt = N^\alpha \int_{-1}^1 J_\alpha(Ny|x|) \tilde{M}_\theta(x', y)\, dy, \tag{2.39}$$

where $\tilde{M}_\theta(x', y)$ is the function (2.12) and

$$J_\alpha(y) = \int_0^1 \rho^{\alpha-1} e^{i\rho y}\, d\rho = (-iy)^{-\alpha} \gamma(\alpha, -iy), \tag{2.40}$$

$\gamma(\alpha, z)$ being the incomplete gamma function ([6], 3.381.1). The representation (2.39) is obtained by passing to polar coordinates and using (1.35).

We note the recursion formula

$$J_\alpha(y) = \frac{1}{(iy)^m} \left[e^{iy} P^\alpha_{m-1}(iy) + (1-\alpha)_m J_{\alpha-m}(y) \right], \qquad m < \alpha, \tag{2.41}$$

obtained by m-fold integration of the integral (2.40) by parts; here $(1-\alpha)_m = (1-\alpha)(2-\alpha) \cdots (-\alpha + m)$, and $P^\alpha_{m-1}(z)$ is the polynomial

$$P^\alpha_{m-1}(z) = z^{m-1} - (\alpha - 1) z^{m-2} + \cdots + (-1)^{m-1} (\alpha - 1) \cdots (\alpha - m + 1)$$
$$= \sum_{k=0}^{m-1} (1-\alpha)_k z^{m-1-k}. \tag{2.42}$$

([2]) For $n = 2$ the second term in (2.35) should be replaced by

$$\frac{\pi}{|x|} \left[\theta\left(\frac{x_2}{|x|}, -\frac{x_1}{|x|} \right) + \theta\left(-\frac{x_2}{|x|}, \frac{x_1}{|x|} \right) \right];$$

similarly in (2.36).

The following recursion relations hold for the polynomials $P_m^\alpha(z)$:

$$P_m^\alpha(z) = z^m + (1-\alpha)P_{m-1}^{\alpha-1}(z), \tag{2.43}$$

$$P_m^\alpha(z) = zP_{m-1}^\alpha(z) + (1-\alpha)_m, \tag{2.44}$$

$$\frac{d}{dz}P_m^{m+1} = mP_{m-1}^m(z). \tag{2.45}$$

Since the symbol $\mathcal{K}_\theta^\alpha(x)$ is homogeneous ($\mathcal{K}_\theta^\alpha(x) = |x|^{-\alpha}\mathcal{K}_\theta^\alpha(x')$), it suffices to consider only its restriction to the unit sphere. Let $m=[\alpha]$, so that $m \leqslant \alpha < m+1$. On the basis of (2.33) and (2.39) we have

$$\mathcal{K}_\theta^\alpha(x') = \lim_{N\to\infty}\int_{-1}^1 A(y)J_\alpha(Ny)\,dy + \sum_{j=1}^{m-1} c_j(N)\frac{\partial^j \tilde{M}_\theta(x',0)}{\partial y^j}, \tag{2.46}$$

where

$$A(y) = A(y,x') = \tilde{M}_\theta(x',y) - \sum_{j=0}^{m-1}\frac{y^j}{j!}\frac{\partial^j\tilde{M}_\theta(x',0)}{\partial y^j} \tag{2.47}$$

and

$$c_j(N) = \frac{N^\alpha}{j!}\int_{-1}^1 y^j J_\alpha(Ny)\,dy, \qquad j=0,1,\dots,m-1. \tag{2.48}$$

The first term in (2.46) will be denoted by

$$I_N(x') = N^\alpha\int_{-1}^1 A(y)J_\alpha(Ny)\,dy. \tag{2.49}$$

Passing to the limit as $N\to\infty$ in (2.46), we get the representation (2.34)–(2.34′) of the theorem. We next separate the cases of integral and nonintegral α.

I. *The case where α is not an integer.*

a) *Asymptotic behavior of the coefficients $c_j(N)$.* Let us show that

$$c_j(N) = \beta_j + \frac{N^\alpha}{j!\,(\alpha-j-1)}\left[Q_j(iN)e^{iN} + (-1)^j Q_j(-iN)e^{-iN}\right] + o(1), \tag{2.50}$$

where

$$\beta_j = \frac{2\Gamma(\alpha)}{j!\,(j-\alpha+1)} i^j \cos\frac{\alpha-j}{2}\pi, \tag{2.51}$$

$$Q_j(z) = z^{-j-1}P_j^{j+1}(z) - z^{-m}P_{m-1}^\alpha(z) = \sum_{\nu=1}^{j}\frac{(-1)_\nu}{z^{\nu+1}} - \sum_{\nu=1}^{m-1}\frac{(1-\alpha)_\nu}{z^{\nu+1}}, \tag{2.52}$$

the $P_j^{j+1}(z)$ being the polynomials in (2.42). We prove (2.50). We have

$$\begin{aligned} c_j(N) &= \frac{N^{\alpha-j-1}}{j!}\int_{-N}^N y^j\,dy\int_0^1 \rho^{\alpha-1}e^{i\rho y}\,d\rho \\ &= \frac{N^{\alpha-j-1}}{j!}\int_0^1 \rho^{\alpha-j-2}\,d\rho\int_{-\rho N}^{\rho N} y^j e^{iy}\,dy = \frac{N^{\alpha-j-1}}{j!}\int_{-N}^N y^j e^{iy}\,dy\int_{|y|/N}^1 \rho^{\alpha-j-2}\,d\rho \\ &= \frac{1}{j!\,(\alpha-j-1)}\left\{N^{\alpha-j-1}\int_{-N}^N y^j e^{iy}\,dy - \int_{-N}^N |y|^{\alpha-1}\operatorname{sgn}^j y e^{iy}\,dy\right\}. \end{aligned} \tag{2.53}$$

The following formulas hold:

$$\int_{-N}^N y^j e^{iy}\,dy = \frac{1}{i^{j+1}}\left[P_j^{j+1}(iN)e^{iN} - P_j^{j+1}(-iN)e^{-iN}\right], \tag{2.54}$$

$$\int_0^N y^{\alpha-1}e^{\pm iy}\,dy = (\mp i)^m\left[N^{\alpha-m}e^{\pm iN}P^{\alpha}_{m-1}(\pm iN) + (1-\alpha)_m\int_0^N y^{\alpha-m-1}e^{\pm iy}\,dy\right], \tag{2.55}$$

as can be proved by induction (the first with the use of (2.43), and the second with the use of (2.44)). With the help of (2.55) we conclude that

$$\int_{-N}^{N}|y|^{\alpha-1}\operatorname{sgn}^j y e^{iy}\,dy = \frac{1}{i^m}\left\{N^{\alpha-m}\left[P^{\alpha}_{m-1}(iN)e^{iN} + (-1)^{m-j}P^{\alpha}_{m-1}(-iN)e^{-iN}\right]\right.$$
$$\left. +(1-\alpha)_m\int_0^N \frac{e^{iy}+(-1)^{m-j}e^{-iy}}{y^{m+1-\alpha}}\,dy\right\}.$$

Substitution of these integrals into (2.53) leads to

$$c_j(N) = \frac{N^{\alpha}}{j!\,(\alpha-j-1)}\left[Q_j(iN)e^{iN} + (-1)^jQ_j(-iN)e^{-iN}\right]$$
$$-\frac{1}{i^m j!\,(\alpha-j-1)}\int_0^N y^{\alpha-m-1}\left[e^{iy}+(-1)^{m-j}e^{-iy}\right]dy.$$

From this it is not hard to get (2.50), taking formulas 3.761.4 and 3.761.9 in [6] into account.

b) *Asymptotic behavior of the integral* $I_N(x')$. The recursion formula (2.41) gives us

$$I_N(x') = \frac{N^{\alpha-m}}{i^m}\int_{-1}^{1}\frac{A(y)}{y^m}P^{\alpha}_{m-1}(iyN)e^{iyN}\,dy$$
$$+\frac{(1-\alpha)_m}{i^m}N^{\alpha-m}\int_{-1}^{1}\frac{A(y)}{y^m}J_{\alpha-m}(Ny)\,dy = \sum_{k=0}^{m-1}I_N^k + I'_N, \tag{2.56}$$

where

$$I_N^k = \frac{(1-\alpha)_k}{i^{k+1}}N^{\alpha-1-k}\int_{-1}^{1}\frac{A(y)}{y^{k+1}}e^{iNy}\,dy \tag{2.57}$$

and

$$I'_N = \frac{(1-\alpha)_m}{i^m}N^{\alpha-m}\int_{-1}^{1}\frac{A(y)}{y^m}J_{\alpha-m}(Ny)\,dy.$$

It will be shown below (see d)) that

$$I'_N \underset{N\to\infty}{\to} \Gamma(\alpha)\int_{-1}^{1}\frac{A(y)}{(-iy)^{\alpha}}\,dy \tag{2.58}$$

and that

$$I_N^{m-1} \underset{N\to\infty}{\to} 0. \tag{2.59}$$

Let us now consider the terms I_N^k for $k = 0, 1, \ldots, m-2$ (the necessity of considering them arises for $m \geqslant 2$). We apply the formula

$$\int_{-1}^{1}f(y)e^{iyN}\,dy = \sum_{\nu=0}^{p}(-1)^{\nu}\frac{f^{(\nu)}(1)e^{iN} - f^{(\nu)}(-1)e^{-iN}}{(iN)^{\nu+1}}$$
$$+\left(\frac{i}{N}\right)^{p+1}\int_{-1}^{1}f^{(p+1)}(y)e^{iNy}\,dy$$

(obtained by integrating by parts $p+1$ times) to I_N^k for the choice $p = m-2-k$. This gives

$$I_N^k = (1-\alpha)_k N^\alpha \sum_{\nu=0}^{m-2-k} (-1)^\nu \frac{A_{k+1}^{(\nu)}(1)e^{iN} - A_{k+1}^{(\nu)}(1)e^{-iN}}{(iN)^{\nu+k+2}}$$
$$+ i^m (1-\alpha)_k N^{\alpha-m} \int_{-1}^{1} A_{k+1}^{(m-k-1)}(y) e^{iNy}\, dy,$$

where $A_{k+1}(y) = A(y)/y^{k+1}$. Below (see d)) it will be shown that

$$N^{\alpha-m} \int_{-1}^{1} A_{k+1}^{(m-k-1)}(y) e^{iNy}\, dy \to 0, \qquad k = 0, 1, \ldots, m-2, \tag{2.60}$$

uniformly in x' as $N \to \infty$. Therefore,

$$\sum_{k=0}^{m-2} I_N^k = N^\alpha e^{iN} \sum_{k=0}^{m-2} (1-\alpha)_k \sum_{\nu=0}^{m-k-2} (-1)^\nu \frac{A_{k+1}^{(\nu)}(1)}{(iN)^{\nu+k+2}}$$
$$- N^\alpha e^{-iN} \sum_{k=0}^{m-2} (1-\alpha)_k \sum_{\nu=0}^{m-k-2} (-1)^\nu \frac{A_{k+1}^{(\nu)}(-1)}{(iN)^{\nu+k+2}} + o(1). \tag{2.61}$$

Let us compute the quantities $A_{k+1}^{(\nu)}(\pm 1)$. Writing $M_j = (\partial^j \tilde{M}_\theta(x', 0))/\partial y^j$, for brevity, we get by Leibniz' formula that

$$A_{k+1}^{(\nu)}(\pm 1) = \sum_{\mu=0}^{\nu} C_\mu^\nu \frac{d^{\nu-\mu}}{dy^{\nu-\mu}} (y^{-1-k}) \bigg|_{y=\pm 1} \cdot \frac{d^\mu}{dy^\mu} \left[\tilde{M}_\theta(x', y) - \sum_{j=0}^{m-1} \frac{M_j}{j!} y^j \right]_{y=\pm 1}.$$

Here

$$\frac{d^\mu}{dy^\mu} [\tilde{M}_\theta(x', y)]\,|_{y=\pm 1} = 0$$

for all $0 \leqslant \mu \leqslant \nu$, if $(n-3)/2 > \nu$. For the latter it suffices that $(n-3)/2 > m-2$, which is true. Therefore, from this (with the equality

$$\frac{d^m y^a}{dy^m} = (-1)^m (-a)_m y^{a-m}$$

taken into account) we get

$$A_{k+1}^{(\nu)}(\pm 1) = (\pm 1)^{k+\nu+1} \sum_{\mu=0}^{\nu} (-1)^{\nu-\mu-1} (k+1)_{\nu-\mu} C_\nu^\mu \sum_{j=\mu}^{m-1} (\pm 1)^j \frac{M_j}{(j-\mu)!}$$
$$= \sum_{j=0}^{m-1} M_j S_{k,\nu,j}^{\pm}, \tag{2.62}$$

where

$$S_{k,\nu,j}^{+} = (-1)^{\nu-1} S_{k,\nu,j}, \qquad S_{k,\nu,j}^{-} = (-1)^{j-k} S_{k,\nu,j}$$

and

$$S_{k,\nu,j} = \sum_{\mu=0}^{\min(\nu,j)} (-1)^\mu C_\nu^\mu \frac{(k+1)_{\nu-\mu}}{(j-\mu)!} = \frac{\nu!}{\mu!} \sum_{\mu=0}^{\min(\nu,j)} (-1)^\mu \frac{(k+\nu-\mu)!}{\mu!\,(\nu-\mu)!\,(j-\mu)!}.$$

Substitution of (2.62) into (2.61) yields

$$\sum_{k=0}^{m-2} I_N^k = -N^\alpha e^{iN} \sum_{j=0}^{m-1} M_j \sum_{k=0}^{m-2} (1-\alpha)_k \sum_{\nu=k}^{m-2} (iN)^{-\nu-2} S_{k,\nu-k,j}$$
$$-N^\alpha e^{-iN} \sum_{j=0}^{m-1} M_j \sum_{k=0}^{m-2} (1-\alpha)_k \sum_{\nu=k}^{m-2} (-1)^{\nu+j} (iN)^{-\nu-2} S_{k,\nu-k,j} + o(1).$$

Hence,

$$\sum_{k=0}^{m-2} I_N^k = -N^\alpha e^{iN} \sum_{j=0}^{m-1} M_j \sum_{\nu=0}^{m-2} s_{\nu,j} (iN)^{-\nu-2}$$
$$-N^\alpha e^{-iN} \sum_{j=0}^{m-1} M_j \sum_{\nu=0}^{m-2} (-1)^{\nu+j} s_{\nu,j} (iN)^{-\nu-2} + o(1), \tag{2.63}$$

where

$$s_{\nu,j} = \sum_{k=0}^{\nu} (1-\alpha)_k S_{k,\nu-k,j}.$$

The representation (2.63) then reduces to the form

$$\sum_{k=0}^{m-2} I_N^k = -N^\alpha \sum_{j=0}^{m-1} M_j \left[\tilde{Q}_j(iN) e^{iN} + (-1)^j \tilde{Q}_j(-iN) e^{-iN}\right] + o(1), \tag{2.63'}$$

where

$$\tilde{Q}_j(z) = \sum_{\nu=0}^{m-2} s_{\nu,j} z^{-\nu-2}.$$

Thus, the behavior of the integral $I_N(x')$ as $N \to \infty$ is determined by (2.58), (2.59), and (2.63′).

c) *Determination of the representation* (2.34). By using the asymptotics (2.50), (2.56)–(2.59), and (2.63′), the representation (2.46) of the symbol $\mathcal{K}_\theta^\alpha(x')$ can be reduced to the form

$$\mathcal{K}_\theta^\alpha(x') = \Gamma(\alpha) \int_{-1}^{1} A(y)(-iy)^{-\alpha}\, dy + \sum_{j=0}^{m-1} \beta_j \frac{\partial^j \tilde{M}_\theta(x',0)}{\partial y^j}$$
$$+ \lim_{N\to\infty} N^\alpha \sum_{j=0}^{m-1} \left\{ \left[\frac{Q_j(iN)}{j!\,(\alpha-j-1)} - \tilde{Q}_j(iN) \right] e^{iN} \right.$$
$$\left. + (-1)^j \left[\frac{Q_j(-iN)}{j!\,(\alpha-j-1)} - \tilde{Q}_j(-iN) \right] e^{-iN} \right\} \frac{\partial^j \tilde{M}_\theta(x',0)}{\partial y^j}. \tag{2.64}$$

Here the first line coincides with the right-hand side of (2.11), multiplied by $\Gamma(\alpha)$, and already gives (2.34). Consequently, it remains to show that the second and third lines in (2.64) disappear. But they contain increasing terms and, therefore, it is natural to expect that

$$\frac{Q_j(z)}{j!\,(\alpha-j-1)} \equiv \tilde{Q}_j(z). \tag{2.65}$$

After showing that (2.65) really holds, we conclude the derivation of (2.34). We have (taking account of the fact that $(-j)_{\nu+1}=0$ for $\nu \geqslant j$) from (2.52) that

$$Q_j(z)=\sum_{\nu=0}^{m-2}[(-j)_{\nu+1}-(1-\alpha)_{\nu+1}]z^{-\nu-2}.$$

Therefore, (2.65) reduces to the form

$$\sum_{k=0}^{\nu}(1-\alpha)_{\nu-k}\frac{k!}{(\nu-k)!}A_{k,j,\nu}=\frac{(-j)_{\nu+1}-(1-\alpha)_{\nu+1}}{j!\,(\alpha-j-1)}, \tag{2.66}$$

where

$$A_{k,j,\nu}=\sum_{\mu=0}^{\min(k,\,j)}(-1)^{\mu}\frac{(\nu-\mu)!}{\mu!\,(j-\mu)!\,(k-\mu)!}. \tag{2.67}$$

Here it is very essential that we were able to find the sum (2.67) in explicit form (see §1.10). Applying (1.50), we reduce the desired relation (2.66) to the form

$$\sum_{k=0}^{\nu}(1-\alpha)_k(-j+k+1)_{\nu-k}=\frac{(1-\alpha)_{\nu+1}-(-j)_{\nu+1}}{j-\alpha+1}.$$

The latter is a particular case of the formula

$$\sum_{k=0}^{\nu}(a)_k(b+k+1)_{\nu-k}=\frac{(a)_{\nu+1}-(b)_{\nu+1}}{a-b},$$

which is easily proved by induction or by direct division of the polynomial $(a)_{\nu+1}-(b)_{\nu+1}$ by the binomial $a-b$. This proves (2.65), and (2.64) becomes (2.34).

d) *Proof of the limits* (2.58), (2.59), *and* (2.60). 1°. For proving (2.58), we have

$$I'_N=\frac{(1-a)_m}{i^m}\left[\int_0^1\frac{A(y)\,dy}{y^\alpha}\int_0^{Ny}\rho^{\alpha-m-1}e^{i\rho}\,d\rho\right.$$
$$\left.+\int_0^1\frac{(-1)^m A(-y)}{y^\alpha}dy\int_0^{Ny}\rho^{\alpha-m-1}e^{-i\rho}\,d\rho\right]. \tag{2.68}$$

Here it is possible to take the limit as $N\to\infty$ under the integral sign by the Lebesgue dominated convergence theorem, in view of the fact that $A(y)|y|^{-\alpha}$ is integrable in a neighborhood of $y=0$ (see Lemma 7). Therefore,

$$\lim_{N\to\infty}I'_N=\frac{(1-\alpha)_m}{i^m}\left[\int_0^1\frac{A(y)}{y^\alpha}dy\int_0^\infty\rho^{\alpha-m-1}e^{i\rho}\,d\rho\right.$$
$$\left.+(-1)^m\int_1^0\frac{A(y\,dy)}{|y|^\alpha}\int_0^\infty\rho^{\alpha-m-1}e^{-i\rho}\,d\rho\right].$$

From this and formulas 3.761.4 and 3.761.9 in [6] we get

$$\lim_{N\to\infty}I'_N=\frac{(1-\alpha)_m}{i^m}\Gamma(\alpha-m)$$
$$\times\left[e^{(\alpha-m)(\pi i/2)}\int_0^1\frac{A(y)\,dy}{y^\alpha}+e^{-(\alpha+m)(\pi i/2)}\int_{-1}^0\frac{A(y)\,dy}{|y|^\alpha}\right]$$
$$=\Gamma(\alpha)\int_{-1}^1\frac{A(y)\,dy}{(-iy)^\alpha},$$

which gives (2.58). Because of the easily proved estimate

$$\left|\int_0^1 A(y)(-iy)^{-\alpha}\,dy\int_{Ny}^{\infty}\rho^{\alpha-m-1}e^{i\rho}\,d\rho\right| \leqslant c\frac{\ln N}{N^{m+1-\alpha}} \tag{2.69}$$

with a constant c not depending on x', the limit (2.58) is uniform in x'.

2°. Observing that (2.59) is (2.60) for $k=m-1$, we shall prove (2.6) for $k=0,1,\ldots,m-1$. It is necessary to show that

$$N^{\alpha-m}\int_0^1 A_{k+1}^{(m-k-1)}(y)e^{iNy}\,dy \underset{N\to\infty}{\to} 0, \qquad k=0,1,\ldots,m-1 \tag{2.70}$$

(the estimates of the integral $N^{\alpha-m}\int_{-1}^{0}$ are analogous). The function $A_{k+1}^{(m-k-1)}(y)$ has, generally speaking, a singularity at $y=0$ and behaves "badly" as $y\to 1$. Lemma 5 suggests a rate of decrease of the integral in (2.70) sufficient to cancel the increasing factor $N^{\alpha-m}$. The crucial point here is the use of (1.48). Let us focus on the singularities at $y=0$ and $y=1$, taking separately $\int_0^{1/2}$ and $\int_{1/2}^{1}$. We apply to $A_{k+1}^{(m-k-1)}(y)$ the representation (1.48), taking $m-1$ instead of m on the basis of Remark 6. This gives

$$A_{k+1}^{(m-k-1)}(y) = \frac{1}{y^2}\int_0^y P_{m-2}\left(\frac{t}{y}\right)\left[\tilde{M}^{(m-1)}(x',t)-\tilde{M}^{(m-1)}(x',y)\right]dt, \tag{2.71}$$

where $P_{m-2}(z)$ is a polynomial of degree $m-2$. The following lemma will be used.

LEMMA 8. *Let $g(t)\in C^{\lambda}[0,a]$, $0<\lambda\leqslant 1$. Then the function*

$$f(y) = \frac{1}{y^{1+\lambda}}\int_0^y b\left(\frac{t}{y}\right)[g(t)-g(y)]\,dt,$$

where $|b(t)|\leqslant c$, satisfies a Hölder condition of the form (1.45) *on* $[0,a]$.

This is easily proved from the equality

$$f(y+h)-f(y) = \left[(y+h)^{-\lambda}-y^{-\lambda}\right]\int_0^1 b(t)[g(yt)-g(t)]\,dt + (y+h)^{-\lambda}$$
$$\times\int_0^1 [g(yt+ht)-g(y+h)-g(yt)+g(y)]b(t)\,dt.$$

We apply this lemma to the function (2.71). Since $\tilde{M}^{(m-1)}(x',y)$ is a Hölder function of order $\lambda_0=\min(1,\lambda-m+1)$ with respect to y for $0\leqslant y\leqslant 1/2$ (uniformly in x'; see Lemma 3), the function (2.71) has, by Lemma 8, the form $A_{k+1}^{(m-k-1)}(y)=y^{\lambda_0-1}f(y)$, where $f(y)$ satisfies condition (1.45) for $0\leqslant y\leqslant 1/2$. But then Lemma 5 (together with Remark 5) asserts that

$$N^{\alpha-m}\left|\int_0^{1/2} A_{k+1}^{(m-k-1)}(y)e^{iNy}\,dy\right| \leqslant c\frac{N^{m-\alpha}}{N^{\lambda_0}} = \frac{c}{N^{\min(m+1-\alpha,\lambda+1-\alpha)}}\to 0. \tag{2.72}$$

Estimation of the remaining integral $N^{\alpha-m}\int_{1/2}^{1}$ in (2.70) reduces to estimation of integrals of the form

$$N^{\alpha-m}\int_{1/2}^{1} y^{j-m}\frac{d^j\tilde{M}_\theta(x',y)}{dy^j}e^{iNy}\,dy, \qquad j=0,1,\ldots,m-1. \tag{2.73}$$

According to (2.12),

$$\frac{d^j\tilde{M}_\theta(x', y)}{dy^j} = |\Sigma_{n-2}| \sum_{\nu=0}^{j} C_j^\nu \frac{\partial^\nu M_\theta(x', y)}{\partial y^\nu} \frac{d^{j-\nu}(1-y^2)^{(n-3)/2}}{dy^{j-\nu}}. \tag{2.74}$$

The means $M_\theta(x', y)$ have, by (1.25), the form

$$M_\theta(x', y) = a\left(x', y, \sqrt{1-y^2}\right),$$

where $a(x', y, z) \in C^\lambda(\Sigma_{n-1} \times [-1, 1] \times [0, 1])$. Proceeding by induction, we get that

$$\frac{\partial^\nu M_\theta(x', y)}{\partial y^\nu} = (1-y^2)^{1/2-\nu} b_\nu\left(x', y, \sqrt{1-y^2}\right),$$

where $b_\nu(x', y, z) \in C^{\lambda-\nu}(\Sigma_{n-1} \times [-1, 1] \times [0, 1])$. Then it is not hard to get from (2.74) that

$$\frac{d^j\tilde{M}_\theta(x', y)}{dy^j} = (1-y^2)^{(n-2)/2-j} c_j\left(x', y, \sqrt{1-y^2}\right), \qquad j = 0, 1, \dots, m-1, \tag{2.75}$$

where $c_j(x', y, z) \in C^{\lambda-j}(\Sigma_{n-1} \times [-1, 1] \times [0, 1])$.

Using our restriction $\alpha < (n+1)/2$, we see that $[\alpha] \leqslant n/2$. But then $(n-2)/2 - j \geqslant 0$ for $j = 0, 1, \dots, m-1$ $(m = [\alpha])$. Therefore, (2.75) can be written in the form

$$\frac{d^j\tilde{M}_\theta(x', y)}{dy^j} = d_j\left(x', y, \sqrt{1-y^2}\right), \qquad j = 0, 1, \dots, m-1,$$

where $d_j(x', y, z) \in C^{\lambda-j}(\Sigma_{n-1} \times [-1, 1] \times [0. 1])$. Consequently, the integral (2.73) has the form

$$N^{\alpha-m}\int_{1/2}^{1} e_j\left(x', y, \sqrt{1-y^2}\right) d^{iNy}\, dy = N^{\alpha-m} e^{iN} \int_0^{1/2} f_j\left(x', y, \sqrt{y}\right) e^{-iN}\, dy, \tag{2.76}$$

where $e_j(x', y, z), f_j(x', y, z) \in C^{\lambda-j}(\Sigma_{n-1} \times [-1, 1] \times [0, 1])$.

Applying Lemma 6 in (2.76), we conclude that

$$N^{\alpha-m}\left|\int_{1/2}^{1} y^{j-m} \frac{\partial^j \tilde{M}_\theta(x', y)}{\partial y^j} e^{iNy}\, dy\right| \leqslant c\frac{N^{\alpha-m}}{N^{\min(1, \lambda-m+1)}} \to 0, \tag{2.77}$$

and an analysis of the proof of Lemma 6 shows that c does not depend on x'.

The estimates (2.72) and (2.77) conclude the proof of the limit (2.70).

II. *The case where α is an integer.* Suppose now that $\alpha = m$ is an integer. Much of what was just gone through remains in force. Therefore, we give only the significantly new points.

a) *Asymptotic behavior of the coefficients $c_j(N)$.* The coefficients $c_j(N)$ in (2.46) have the same asymptotics (2.50) as before for $j = 0, 1, \dots, m-2$. But in the case $j = m-1$, (2.53) is replaced by

$$c_{m-1}(N) = \frac{1}{(m-1)!}\int_{-N}^{N} y^{m-1} \ln \frac{N}{|y|} e^{iy}\, dy. \tag{2.78}$$

Let us determine the asymptotic behavior of this integral. From (2.55) we get that

$$\int_0^z y^{m-1}e^{iy}\,dy = \frac{e^{iz}}{i^m}P^m_{m-1}(iz) + i^m(m-1)!, \tag{2.79}$$

where $P^m_{m-1}(z)$ is the polynomial in (2.42). Therefore, integrating in (2.78) and using (2.79), we get

$$\begin{aligned} C_{m-1}(N) &= \frac{1}{i^m(m-1)!}\int_{-N}^{N}\frac{P_{m-1}(iy)}{y}e^{iy}\,dy \\ &= \frac{(-1)^{m-1}}{i^m}\sum_{\nu=0}^{m-1}\frac{(-i)^\nu}{\nu!}\int_{-N}^{N}y^{\nu-1}e^{iy}\,dy. \end{aligned}$$

Hence, by (2.54),

$$\begin{aligned} c_{m-1}(N) &= i^m\sum_{\nu=0}^{m-2}\frac{(-1)^\nu}{(\nu+1)!}\left[P_\nu^{\nu+1}(iN)e^{iN} - P_\nu^{\nu+1}(-iN)e^{-iN}\right] \\ &\quad + 2i^{m-1}\int_0^N\frac{\sin y}{y}\,dy, \end{aligned}$$

and then

$$c_{m-1}(N) = i^m\left[e^{iN}R_{m-2}(iN) - e^{-iN}R_{m-2}(-iN)\right] + \pi i^{m-1} + o(1), \tag{2.80}$$

where

$$R_{m-2}(z) = \sum_{\nu=0}^{m-2}\frac{(-1)^\nu}{(\nu+1)!}P_\nu^{\nu+1}(z) = \sum_{\mu=0}^{m-2}\left(\frac{(-1)^\mu}{\mu!}\sum_{s=1+\mu}^{m-1}\frac{1}{s}\right)z^\mu. \tag{2.81}$$

b) *Asymptotic behavior of the integral* $I_N(x')$. The representation (2.56) has order smaller by 1:

$$\begin{aligned} I_N(x') &= \frac{N}{i^{m-1}}\int_{-1}^{1}A(y)y^{-m}P^\alpha_{m-2}(iyN)e^{iyN}\,dy \\ &\quad + \frac{(1-m)_{m-1}}{i^{m-1}}N\int_{-1}^{1}A(y)y^{-m+1}J_1(Ny)\,dy = \sum_{k=0}^{m-2}I_N^k + I_N', \end{aligned} \tag{2.82}$$

where I_N^k is the integral (2.57) with $\alpha = m$, and

$$\begin{aligned} I_N' &= i^{m-1}(m-1)!N\int_{-1}^{1}A(y)y^{-m+1}\,dy\int_0^1 e^{i\rho yN}\,d\rho \\ &= (m-1)!\int_{-1}^{1}A(y)(-iy)^m(1-e^{iyN})\,dy \\ &= \Gamma(m)\int_{-1}^{1}\frac{A(y)\,dy}{(-iy)^m} + o(1). \end{aligned} \tag{2.83}$$

As for the sum $\sum_{k=0}^{m-2}I_N^k$, its asymptotic behavior does not change and is given by (2.63′).

c) *Determination of the representation* (2.34′). On the basis of the asymptotics expressed in a) and b) we now get, from (2.46), the following representation in place of (2.64):

$$\mathcal{K}_\theta^\alpha(x') = \Gamma(\alpha)\int_{-1}^{1}\frac{A(y)}{(-iy)^\alpha}dy + \sum_{j=0}^{m-2}\beta_j\frac{\partial^j\tilde{M}_\theta(x',0)}{\partial y^j} + \pi i^{m-1}\frac{\partial^{m-1}\tilde{M}_\theta(x',0)}{\partial y^{m-1}}$$
$$+\lim_{N\to\infty}\{e^{iN}[i^m R_{m-2}(iN) - N^m\tilde{Q}_{m-1}(iN)]$$
$$-e^{-iN}[i^m R_{m-2}(-iN) - (-N)^m\tilde{Q}_{m-1}(-iN)]\}\frac{\partial^{m-1}\tilde{M}_\theta(x',0)}{\partial y^{m-1}}, \tag{2.84}$$

where the $\tilde{Q}_{m-1}(z)$ are the same functions as in (2.64), and the $R_{m-2}(z)$ are the polynomials in (2.81).

Let us show that $z^m\tilde{Q}_{m-1}(iz) \equiv i^m R_{m-2}(iz)$, i.e., that

$$z^m\tilde{Q}_{m-1}(z) = (-1)^m R_{m-2}(z). \tag{2.85}$$

Computing the coefficients, we arrive at a system of equalities equivalent to (2.85):

$$\sum_{k=0}^{\nu}(-1)^{\nu-k}\frac{(m-1)!k!}{(m-\nu+k-1)!(\nu-k)!}A_{k,j,\nu} = \frac{(-1)^\nu}{(m-2-\nu)!}\sum_{k=m-\nu-1}^{m-1}\frac{1}{s},$$
$$\nu = 0, 1, \ldots, m-2.$$

If the $A_{k,j,\nu}$ from (1.50) are substituted here, the left-hand side reduces to the form

$$(-1)^\nu\sum_{k=0}^{\nu}\frac{(m-1-\nu)(m-\nu)\cdots(m-2+k-\nu)}{(m-\nu+k-1)!}$$
$$= \frac{(-1)^\nu}{(m-2-\nu)!}\sum_{k=0}^{\nu}\frac{1}{m-1+k-\nu},$$

which is what was required.

By (2.85) and (2.84), we get finally that

$$\mathcal{K}_\theta^\alpha(x') = \Gamma(\alpha)\int_{-1}^{1}\frac{\tilde{M}_\theta(x',y) - \sum_{j=0}^{\alpha-1}(y^j)/j!\,(\partial^j\tilde{M}_\theta(x',0))/\partial y^j}{(-iy)^\alpha}dy$$
$$+2\Gamma(\alpha)\sum_{j=0}^{\alpha-2}\frac{t^j\cos(\alpha-j)/2\pi}{j!(j-\alpha+1)}\frac{\partial^j\tilde{M}_\theta(x',0)}{\partial y^j} + \pi i^{\alpha-1}\frac{\partial^{\alpha-1}\tilde{M}_\theta(x',0)}{\partial y^{\alpha-1}},$$

which, by (2.11) coincides with (2.34′). Theorem 4 is completely proved.

REMARK 11. Recall that the symbol $\mathcal{K}_\theta^\alpha(x)$ of the potential (2.1) is given by a convergent integral (2.2) only when $0 < \alpha < (n+1)/2$. Everywhere below, the symbol $\mathcal{K}_\theta^\alpha(x)$ is understood for $(n+1)/2 \leqslant \alpha$ to be the right-hand side of the representation (2.34)–(2.34′).

COROLLARY 1. *The symbol* $\mathcal{K}_\theta^\alpha(x)$, $x \neq 0$, *of a generalized Riesz potential is an analytic function of the parameter* α *in the strip* $0 < \operatorname{Re}\alpha < \lambda + 1$.

This corollary follows from the representations (2.34)–(2.34′) and the corollary to Theorem 2 (see (2.15)!).

COROLLARY 2. *The symbol* $\mathcal{K}_\theta^\alpha(x)$, $x \neq 0$, *of a generalized Riesz potential has order of smoothness less by* $\alpha - 1$ *than that of the characteristic* $\theta(\sigma)$ *of the potential. Namely,*

$$\theta(\sigma) \in C^\lambda(\Sigma_{n-1}) \Rightarrow \mathcal{K}_\theta^\alpha(x') \in C^{\lambda-\alpha+1}(\Sigma_{n-1}), \tag{2.86}$$

where $0 < \alpha < \lambda + 1$, $\lambda > 0$, $\alpha \neq 1, 2, 3, \ldots$, *and* $\lambda - \alpha \neq 0, 1, 2, \ldots$. *But if* α *or* $\lambda - \alpha$ *is an integer, then* $\mathcal{K}_\theta^\alpha(x') \in C_*^{\lambda+1-\alpha}(\Sigma_{n-1})$. *It is possible to take* $\lambda = 0$ *in the case* $0 < \alpha < 1$.

Indeed, it suffices to apply Theorem 3 and Lemma 2 to the right-hand sides in (2.34) and (2.34′).

REMARK 12. If α is not an integer, then the symbol $\mathcal{K}_\theta^\alpha(x)$ can also be written in the form

$$\mathcal{K}_\theta^\alpha(x) = \Gamma(\alpha)\cos\frac{\alpha\pi}{2}\ \text{f.p.}\int_{\Sigma_{n-1}} \frac{\theta(\sigma)\,d\sigma}{|\sigma\cdot x|^\alpha} + i\Gamma(\alpha)\sin\frac{\alpha\pi}{2}\ \text{f.p.}\int_{\Sigma_{n-1}} \frac{\theta(\sigma)\,d\sigma}{(\sigma\cdot x)^\alpha}, \tag{2.87}$$

where $(\sigma\cdot x)^\alpha = |\sigma\cdot x|^\alpha \operatorname{sgn}(\sigma\cdot x)$, and the f.p.-constructions in (2.87) are defined similarly to Definition 5. Moreover,

$$\mathcal{K}_\theta^\alpha(x) = \frac{\pi i^{\alpha-1}}{|x|^\alpha}\frac{\partial^{\alpha-1}\tilde{M}_\theta(x',0)}{\partial y^{\alpha-1}} \tag{2.88}$$

in the case of 1) an even characteristic $\theta(\sigma)$ and an odd integer $\alpha = 1, 3, 5, \ldots$ and 2) an odd characteristic $\theta(\sigma)$ and an even $\alpha = 2, 4, 6, \ldots$.

REMARK 13. In view of (2.37), we proved in Theorem 4 that

$$\lim_{N\to\infty}\int_{\Sigma_{n-1}}\theta(\sigma)\,d\sigma\int_0^N e^{i\rho\sigma\cdot x'}\rho^{\alpha-1}\,d\rho = \Gamma(\alpha)\ \text{f.p.}\int_{\Sigma_{n-1}}\frac{\theta(\sigma)\,d\sigma}{(-ix'\cdot\sigma)^\alpha}$$

for nonintegral $\alpha \neq 1, 2, 3, \ldots$, and this limit is uniform in x'; for $\alpha = 1, 2, 3, \ldots$ the right-hand side should be replaced by the right-hand side of (2.34′).

4. On the justification for passing to the symbol of a potential. The function $\mathcal{K}_\theta^\alpha(x)$, $0 < \alpha < n$, which was constructed by the rule (2.34)–(2.34′) and called the *symbol* of the potential, coincides with the (conditionally convergent) Fourier transform of the kernel of the potential for $0 < \alpha < (n+1)/2$. We want to see that for all $0 < \alpha < n$ the function $\mathcal{K}_\theta^\alpha(x)$ coincides with the Fourier transform of the kernel, understood in the sense of S' distributions. We must show that

$$(K_\theta^\alpha\varphi)^\wedge = \mathcal{K}_\theta^\alpha(x)\hat{\varphi}(x). \tag{2.89}$$

Since $\mathcal{K}_\theta^\alpha(x)$ is not a multiplier in S (and not even in the Lizorkin space $\Psi = \hat{\Phi}$ (see [8] and [9]), which is more suited for these purposes, if $\theta(\sigma) \notin C^\infty(\Sigma_{n-1})$), the Gel'fand-Shilov theorem ([5], Chapter III, §3.7, Theorem 1) is not applicable, and it is necessary to justify the transition (2.89).

THEOREM 5. *Suppose that* $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda > \max(0, \alpha - 1)$, $0 < \alpha < n$. *Then* (2.89) *holds*:

$$\frac{1}{(2\pi)^n}\int_{R^n} e^{-ix\cdot t}\mathcal{K}_\theta^\alpha(t)\hat{\varphi}(t)\,dt = \int_{R^n}\frac{\theta(t')}{|t|^{n-\alpha}}\varphi(x-t)\,dt \tag{2.90}$$

for all $\varphi(t) \in S(R^n)$.(3)

PROOF. Both sides of (2.90) are analytic in α for $0 < \operatorname{Re}\alpha < n$. (For the left-hand side this can be verified directly with the help of Corollary 1 to Theorem 4 and the homogeneity of the symbol $\mathcal{K}_\theta^\alpha(t)$, and for the right-hand side the analyticity is obvious.) Therefore, it suffices to prove (2.90) for $0 < \alpha < 1$. We reduce (2.90) to the form

$$\int_{R^n} e^{-ix\cdot t}\mathcal{K}_\theta^\alpha(t)\hat{\varphi}(t)\,dt = \int_0^\infty \rho^{\alpha-1}\,d\rho\int_{\Sigma_{n-1}}\theta(\sigma)\,d\sigma\int_{R^n} e^{is(\rho\sigma-x)}\hat{\varphi}(s)\,ds$$

or

$$\int_{R^n} e^{-ix\cdot t}\mathcal{K}_\theta^\alpha(t)\hat{\varphi}(t)\,dt = \lim_{N\to\infty}\int_{R^n} e^{-ix\cdot t}\hat{\varphi}(t)\,dt\int_{\Sigma_{n-1}}\theta(\sigma)\,d\sigma\int_0^N \rho^{\alpha-1}e^{i\rho\sigma\cdot t}\,d\rho. \tag{2.91}$$

It remains to refer to (2.37). The Lebesgue dominated convergence theorem allows us to pass to the limit under the integral sign in (2.91) (see (2.38)).

COROLLARY. *The symbol* $\mathcal{K}_\theta^\alpha(x)$ *coincides with the generalized* (*in the sense of the* S' *distributions*) *Fourier transform of the kernel* $k_\theta^\alpha(x) = \theta(x')|x|^{\alpha-n}$ *for all* $0 < \alpha < n$.

§3. Some general considerations about inversion of potentials

In the simplest case of a Riesz potential K^α (i.e., $\theta \equiv \text{const}$) the inverse operator was constructed by us in [25] with the aid of so-called hypersingular integrals (HSI's). In the general case, when $\theta \not\equiv \text{const}$, the construction of the inverse operator is closely connected, in view of (2.89), with the properties of the function $\mathcal{K}_\theta^\alpha(x)$. In studying the operator in, for example, $L_p(R^n)$, we shall naturally be interested in how much the image $K_\theta^\alpha(L_p)$ differs from the image $K^\alpha(L_p)$ of the Riesz potential. (Since $(K^\alpha\varphi)\hat{} = |x|^{-\alpha}\hat{\varphi}(x)$, while $(K_\theta^\alpha\varphi)\hat{} = \mathcal{K}_\theta^\alpha(x')|x|^{-\alpha}\hat{\varphi}(x)$, the restriction of the symbol $\mathcal{K}_\theta^\alpha(x)$ to the unit sphere could be called the "characteristic of the image distortion.") Let us begin with the following obvious statement:

If $\mathcal{K}_\theta^\alpha(x/|x|)$ *is a p-multiplier, then* $K_\theta^\alpha(L_p) \subseteq K^\alpha(L_p)$. *If, moreover,* $1/\mathcal{K}_\theta^\alpha(x/|x|)$ *is also a p-multiplier, then* $K_\theta^\alpha(L_p) = K^\alpha(L_p)$.

On the basis of the study of $\mathcal{K}_\theta^\alpha(x)$ carried out in §2, this leads to the following theorem.

THEOREM 6. *If* $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda \geqslant \alpha + n - 1$, *then, always,* $K_\theta^\alpha(L_p) \subseteq K^\alpha(L_p)$. *If, moreover,*

$$\mathcal{K}_\theta^\alpha(x') \neq 0, \qquad x' \in \Sigma_{n-1}, \tag{3.1}$$

then $K_\theta^\alpha(L_p) = K^\alpha(L_p)$.

(3) For the validity of (2.90) it actually suffices that $\varphi(x) \in L_2(R^n) \cap L_\infty(R^n)$ and that $|\varphi(x)| \leqslant c|x|^{-a}$, $a > \alpha$, and $|\hat{\varphi}(x)| \leqslant c|x|^{-b}$, $b > n - \alpha$, as $|x| \to \infty$.

PROOF. By Corollary 2 to Theorem 4, $\mathcal{K}_\theta^\alpha(x') \in C^{\lambda-\alpha+1}(\Sigma_{n-1})$, and $\lambda - \alpha + 1 \geqslant n$ here. But then a direct check of the conditions in the well-known theorem of Mikhlin on p-multipliers shows that $\mathcal{K}_\theta^\alpha(x')$ is a p-multiplier. If $\mathcal{K}_\theta^\alpha(x') \neq 0$, then $1/\mathcal{K}_\theta^\alpha(x')$ is also a p-multiplier, by the same theorem of Mikhlin. Theorem 6 then follows from the above statement.

REMARK 14. In the case $p = 2$ the condition $\lambda \geqslant \alpha + n - 1$ in Theorem 6 can be relaxed to $\lambda > \max(0, \alpha - 1)$.

When the symbol of the potential is nonsingular on the unit sphere, we shall call the case (3.1) *elliptic*. Clearly, in the elliptic case the inverse operator $(K_\theta^\alpha)^{-1}$ can be constructed in the form $(K_\theta^\alpha)^{-1} = A\mathbf{D}^\alpha = \mathbf{D}^\alpha A$, where A is the multidimensional singular convolution operator having $1/K_\theta^\alpha(x')$ as its symbol, and $\mathbf{D}^\alpha$ is the hypersingular Riesz differentiation operator (the operator inverse to the Riesz potential K^α; see [25]). Of considerably greater interest and substance here is the question of constructing the inverse operator directly in the form of a hypersingular integral (see (3)) with some characteristic $\Omega(t')$. This problem, which includes the explicit construction of the characteristic $\Omega(t')$ from the characteristic $\theta(t')$, will be completely solved in §§5 and 6. In §4 we shall first consider hypersingular constructions.

We remark that in the nonelliptic case, when condition (3.1) is violated, the inverse operator no longer has the form of the HS construction (3). The form of the inverse operator will be determined each time by the specific nature of the symbol's singularity. One such case, where the characteristic $\theta(\sigma)$ of the potential is linear, $\theta(\sigma) = \sigma \cdot a = \sigma_1 a_1 + \cdots + \sigma_n a_n$, was considered by us in [26]. Then the symbol $\mathcal{K}_\theta^\alpha(x)$ has the form

$$\mathcal{K}_\theta^\alpha(x) = \text{const}\,|x|^{-\alpha}\,(x' \cdot a)$$

and is a fortiori singular on the unit sphere (on the section of it by the hyperplane $x \cdot a = 0$). This case involves the construction of an inverse operator of the form

$$\varphi(x) = \text{const} \int_0^\infty d\xi \int_{R^n} \frac{(\Delta_t^l f)(x - a\xi)}{|t|^{n+\alpha+1}}\,dt \tag{3.2}$$

with a certain "exotic" nature of convergence. (Unlike in the case of the usual HSI (3), which was truncated by excising a shrinking ball $|t| < \varepsilon$, in (3.2) the appropriate truncation here is achieved by excising the cylinder $|t| < \varepsilon$ in R_+^{n+1} and cutting off the half-space $\xi > N$, with the convergence as $\varepsilon \to 0$ realized in the L_p-norm and the convergence as $N \to \infty$ realized as convergence in measure.) In the conclusion of this section we give one more example of a characteristic $\theta(\sigma)$ for which the image $K_\theta^\alpha(L_p)$ is different from $K^\alpha(L_p)$ and for which, consequently, the construction of the inverse operator $(K_\theta^\alpha)^{-1}$ does not have the form (3). Namely, let us consider the most elementary nonsmooth (discontinuous on a coordinate plane) characteristic

$$\theta(x') = \operatorname{sgn} x_j, \qquad j = 1, 2, \ldots, n. \tag{3.3}$$

In this case

$$\mathcal{K}_\theta^\alpha(x) = \frac{i\mu_n(\alpha) x_j}{\left(|x|^2 - x_j^2\right)^{(\alpha-1)/2}|x|^2} F\left(1, 1 - \frac{1}{2}; \frac{3}{2}; \frac{x_j^2}{|x|^2}\right), \tag{3.4}$$

where

$$\mu_n(\alpha) = 2^{1+\alpha}\pi^{(n-1)/2}\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma^{-1}\left(\frac{n-\alpha}{2}\right),$$

and F is the hypergeometric function of Gauss.

Indeed, we have (restricting ourselves to formal expressions)

$$\mathcal{K}_\theta^\alpha(x) = F\left(\frac{\operatorname{sgn} t_j}{|t|^{n-\alpha}}\right) = \frac{1}{(2\pi)^n}(\operatorname{sgn} t_j)^\wedge * \frac{\gamma_n(\alpha)}{|t|^\alpha}$$
$$= \frac{\gamma_n(\alpha)}{(2\pi)^n}\frac{1}{|t|^\alpha} * \left[(\operatorname{sgn} t_j)^\wedge \otimes (1(\tilde{t}))^\wedge\right],$$

where $\tilde{t} = (t_1,\dots,t_{j-1}, t_{j+1},\dots,t_n)$. Since $(\operatorname{sgn} t_j)^\wedge = 2i/t_j$ and $(1(\tilde{t}))^\wedge = (2\pi)^{n-1}\delta(\tilde{t})$, it follows that

$$\mathcal{K}_\theta^\alpha(x) = \frac{i\gamma_n(\alpha)}{\pi}\frac{1}{|t|^\alpha} * \left[\frac{1}{t_j}\otimes\delta(\tilde{t})\right] = \frac{i\gamma_n(\alpha)}{\pi}\int_{R^n}\frac{\delta(\tilde{t})\,dt}{t_j|t-x|^\alpha}$$
$$= \frac{i\gamma_n(\alpha)}{\pi}\int_{-\infty}^{\infty}\frac{dt_j}{t_j\left[(t_j-x_j)^2+|\tilde{x}|^2\right]^{\alpha/2}}$$
$$= -\frac{i\gamma_n(\alpha)}{\pi}\int_{-\infty}^{\infty}\frac{d\xi}{(\xi-x_j)(\xi^2+\rho^2)^{\alpha/2}}, \tag{3.5}$$

where $\rho^2 = |x|^2 - x_j^2$.

It is possible to express the singular integral thus obtained in terms of a hypergeometric function:

$$\int_{-\infty}^{\infty}\frac{dt}{(t-x)(t^2+\rho^2)^{\alpha/2}} = -\frac{2\sqrt{\pi}\,\Gamma((1+\alpha)/2)}{\Gamma(\alpha/2)}\frac{x\rho^{1-\alpha}}{x^2+\rho^2}$$
$$\times F\left(1, 1-\frac{\alpha}{2}; \frac{3}{2}; \frac{x^2}{x^2+\rho^2}\right), \tag{3.6}$$

which takes (3.5) into (3.4).

For an even integer α the hypergeometric function in (3.6) can be expressed as a Jacobi polynomial

$$P_{(\alpha-1)/2}^{(1/2,(1-\alpha)/2)}\left(\frac{1}{2}\frac{x^2}{x^2+\rho^2}\right);$$

see [6], 8.962.1. In the case of odd $\alpha = 2m+1$ it can be shown that

$$F\left(1, \frac{1}{2}-m; \frac{3}{2}; z\right) = \frac{1}{2}\frac{(2m-1)!!}{(2m)!!}\frac{(1-z)^m}{\sqrt{z}}\ln\frac{1+\sqrt{z}}{1-\sqrt{z}} + P_{m-1}(z), \tag{3.7}$$

where $P_{m-1}(z) = \sum_{k=1}^m a_{m,k} z^{k-1}(1-z)^{m-k}$ is a polynomial of degree $m-1$ with coefficients

$$a_{m,k} = \frac{(2m-1)!!(k-1)!}{2^{m-k-1}m!}\sum_{j=k}^{m}(-1)^{k-j}\frac{(2j-2k-1)!!}{(2j-1)!!}C_m^j.$$

(The formulas (3.6) and (3.7) are new, as far as the author knows.) The symbol (3.4) can thereby be computed in terms of elementary functions when α is an integer. In particular,

$$\mathcal{K}_\theta^\alpha(x) = \frac{2\pi^{(n-1)/2} i}{\Gamma((n-1)/2)} \frac{\operatorname{sgn} x_j}{|x|} \ln \frac{|x|^2 - x_j^2}{(|x| + x_j)^2},$$

$$\mathcal{K}_\theta^\alpha(x) = \frac{4\pi^{n/2}}{\Gamma((n-2)/2)} \frac{i x_j}{|x|^2 \sqrt{|x|^2 - x_j^2}}$$

for $\alpha = 1$ and $\alpha = 2$, respectively.

It follows from (3.4) that the symbol $\mathcal{K}_\theta^\alpha(x')$ vanishes at the points of the hyperplane $x_j = 0$ and becomes infinite at two points of the sphere. Consequently, *for any* α *the image of the potential* K_θ^α *with characteristic* $\theta(x) = \operatorname{sgn} x_j$ *does not coincide with the image of the Riesz potential* (an essential difference from the one-dimensional case; see [23], §5).

§4. Hypersingular integrals with homogeneous characteristics

Multidimensional hypersingular integrals (HSI's) were apparently first used by Stein ([33], pp. 161–162) in describing the space of Bessel potentials of order $0 < \alpha < 2$. An extension of HSI's to values $\alpha \geqslant 2$ can be made either in terms of the finite part of the integral (regularization of the generalized function $r^{-n-\alpha}$) by subtraction of a partial sum of the Taylor series, or by taking finite differences. The latter approach is preferable in some respects (although it is equivalent, generally speaking, to the first; concerning this in the case of smooth functions see subsection 4) and was used by Lizorkin [10], who introduced HSI's of the form

$$\int_{R^n} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}}\, dt \tag{4.1}$$

with the central difference $(\Delta_t^l f)(x)$ and obtained a description of the spaces of Bessel potentials in terms of them in the general case (the construction (4.1) can be called the multidimensional analogue of the fractional derivative of Marchaud). More generally, hypersingular integrals are defined to be integrals of the form

$$\int_{R^n} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}} \Omega(x, t)\, dt. \tag{4.2}$$

There are a number of references ([38], [44]–[49]) in which the integrals (4.1), (4.2) and integrals similar to them with a Taylor series remainder instead of a finite difference have been studied in the framework of spaces of Bessel potentials. In [24], [25], and [29] the author used the constructions (4.1) to introduce new spaces $L_{p,r}^\alpha(R^n)$ of potentials of Riesz type. The action of the operators (4.2) was also studied in [24] and [29] in the framework of these spaces. We mention, in addition, the papers [12]–[15], in which HSI's of order $0 < \alpha < 1$ were examined from the point of view of their inclusion in the class of pseudodifferential operators.

The function $\Omega(x, t)$ will be called the *characteristic* of the HSI (4.2). Here we shall be interested in the case of a characteristic homogeneous in t and not dependent on x: $\Omega = \Omega(t/|t|)$. Let

$$(\mathbf{D}_\Omega^\alpha f)(x) = \frac{1}{d_{n,l}(\alpha)} \int_{R^n} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}} \Omega\left(\frac{t}{|t|}\right) dt, \qquad \alpha > 0; \tag{4.3}$$

the choice of the normalizing factor $d_{n,l}(\alpha)$ is indicated below in subsection 2.

In this section we present without proofs some results needed in what follows for HSI's with homogeneous characteristics. Most of them were obtained by the author in [32], where proofs can be found.

The notation $(\Delta_t^l f)(x)$ for the finite difference of a function $f(x)$ of order l with step $t \in R^n$ will be used both for the noncentered difference

$$(\Delta_t^l f)(x) = \sum_{k=0}^{l} (-1)^k C_l^k f(x - kt),$$

and for the centered difference

$$(\Delta_t^l f)(x) = \sum_{k=0}^{l} (-1)^k C_l^k f(x + (l/2 - k)t).$$

1. Classification of hypersingular integrals. We introduce the following concept.

DEFINITION 6. The integral (4.3) will be called an HSI of *neutral type* if it is constructed with the help of a noncentered difference, and an HSI of *even* (*odd*) *type* if it is constructed with the help of a centered difference or even (odd) order l.

The integral (4.3) converges (on sufficiently smooth functions) for

$$l > \alpha.$$

In the case of an HSI of neutral type and with an even characteristic ($\Omega(t') = \Omega(-t')$) the order of the differences can be lowered, because of the conditional convergence, to

$$l > 2[\alpha/2] \tag{4.4}$$

with the obligatory choice $l = \alpha$ for $\alpha = 1, 3, 5, \ldots$ (see [25], where $\Omega(t') \equiv 1$). It is assumed everywhere below that l is so chosen.

As explained below, the neutral type of HSI will have certain advantages over the even and odd types (this has already manifested itself in the possibility (4.4) of lowering the order l). It will be more universal in problems of inverting potentials. In particular, it makes sense to consider an HSI of even (odd) type only for even (odd) characteristics $\Omega(t')$. Namely, if the characteristic is arbitrary, and

$$\Omega(x') = \Omega_+(x') + \Omega_-(x'), \qquad \Omega_\pm(x') = \frac{\Omega(x') \pm \Omega(-x')}{2},$$

then

$$\mathbf{D}_\Omega^\alpha f \equiv \mathbf{D}_{\Omega_+}^\alpha f, \qquad \mathbf{D}_\Omega^\alpha f \equiv \mathbf{D}_{\Omega_-}^\alpha f \tag{4.5}$$

for integrals of even and odd type, respectively (integrals of even or odd type are annihilated in the case of a characteristic of the opposite parity). The relations (4.5) follow, for example, from (4.13) and (4.14). On the other hand, an HSI of neutral type has its own "peculiarities," which are reflected in the following remark.

REMARK 15. In the case $\alpha = 1, 3, 5, \ldots$ an HSI of neutral type is identically annihilated when $l > \alpha$:

$$\int_{R^n} |t|^{-n-\alpha} (\Delta_t^l f)(x) \Omega\left(\frac{t}{|t|}\right) dt = 0,$$

for any f and Ω (details on this can be found in [25] for the case $\Omega \equiv 1$). But if $l = \alpha$, then it converges (conditionally) if and only if $\Omega(t/|t|)$ is even.

2. The normalization constants $d_{n,l}(\alpha)$. These constants are chosen in such a way that $F(\mathbf{D}_\Omega^\alpha f) = |x|^\alpha \hat{f}(x)$ for $\Omega(t') \equiv 1$. This turns out to be possible also for HSI's of neutral and even types. But in the case of odd type $\mathbf{D}_\Omega^\alpha f \equiv 0$ for $\Omega(t') = 1$; therefore, for an HSI of odd type the constant $d_{n,l}(\alpha)$ will be chosen from considerations of symmetry and analyticity.

We introduce the following functions of the parameter α:

$$A_l(\alpha) = \begin{cases} A_l'(\alpha) = \sum_{k=0}^{l} (-1)^{k-1} C_l^k k^\alpha & \text{for a noncentered difference,} \\ A_l''(\alpha) = 2 \sum_{k=0}^{[l/2]} (-1)^{k-1} C_l^k \left(\frac{l}{2} - k\right)^\alpha & \text{for a centered difference.} \end{cases} \tag{4.6}$$

Furthermore, it is possible to write

$$A_l''(\alpha) = \begin{cases} \sum_{k=0}^{l} (-1)^k C_l^k \left|\frac{l}{2} - k\right|^\alpha & \text{for even } l, \\ \sum_{k=0}^{l} (-1)^k C_l^k \left|\frac{l}{2} - k\right|^\alpha \operatorname{sgn}\left(\frac{l}{2} - k\right) & \text{for odd } l. \end{cases}$$

LEMMA 9. *The zeros of the function $A_l'(\alpha)$ are the integers $\alpha = 1, 2, \ldots, l-1$, and those of the function $A_l''(\alpha)$ are the even integers $\alpha = 2, 4, \ldots, l-2$ for even l and the odd integers $\alpha = 1, 3, \ldots, l-2$ for odd l.*

THEOREM 7. *For an HSI of neutral or even type the normalization constants $d_{n,l}(\alpha)$ are analytic functions of the parameter α and can be computed by the formulas*

$$d_{n,l}(\alpha) = \beta_n(\alpha) \frac{A_l(\alpha)}{\sin(\alpha\pi/2)}, \tag{4.7}$$

where

$$\beta_n(\alpha) = \frac{\pi^{(n/2+1)}}{2^\alpha \Gamma(1+\alpha/2) \Gamma((n+\alpha)/2)}. \tag{4.8}$$

We remark that for an even integer α the expression $A_l(\alpha)/\sin(\alpha\pi/2)$ is understood according to Lemma 9 as

$$\lim_{\xi \to \alpha} \frac{A_l(\xi)}{\sin(\xi\pi/2)} = \frac{2}{\pi} (-1)^{\alpha/2} \frac{d}{d\alpha} A_l(\alpha).$$

For an HSI of odd type we start from (4.7) and, taking Lemma 9 into account, define

$$d_{n,l}(\alpha) = \beta_n(\alpha)\frac{A_l''(\alpha)}{\cos(\alpha\pi/2)}. \tag{4.9}$$

3. The symbol of a hypersingular integral. In terms of Fourier transforms we have

$$(\mathbf{D}_\Omega^\alpha f)\hat{} = \mathcal{D}_\Omega^\alpha(x)\hat{f}(x), \tag{4.10}$$

where $\mathcal{D}_\Omega^\alpha(x)$ is the symbol of the HSI. It has the form

$$\mathcal{D}_\Omega^\alpha(x) = \begin{cases} \dfrac{1}{d_{n,l}(\alpha)}\displaystyle\int_{R^n}\frac{(1-e^{ix\cdot t})^l}{|t|^{n+\alpha}}\Omega(t')\,dt & \text{for a noncentered difference,} \\ \dfrac{1}{d_{n,l}(\alpha)}\displaystyle\int_{R^n}\frac{(e^{(ix\cdot t)/2}-e^{-(ix\cdot t)/2})^l}{|t|^{n+\alpha}}\Omega(t')\,dt & \text{for a centered difference.} \end{cases} \tag{4.11}$$

Formulas (4.10) and (4.11) are easy to prove for sufficiently nice functions $f(x)$ (see [25], Appendix, where $\Omega \equiv 1$). Obviously, the symbol is homogeneous of degree α.

THEOREM 8. *Let $\Omega(t') \in L_1(\Sigma_{n-1})$. The following representations of the symbol $\mathcal{D}_\Omega^\alpha(x)$ by surface integrals are valid:*(4)

$$\mathcal{D}_\Omega^\alpha(x) = \frac{\Gamma((n+\alpha)/2)}{2\pi^{(n-1)/2}\Gamma((1+\alpha)/2)\cos(\alpha\pi/2)}\int_{\Sigma_{n-1}}\Omega(\sigma)(-ix\cdot\sigma)^\alpha\,d\sigma,$$

$$\alpha \neq 1,3,5,\ldots, \tag{4.12}$$

$$\mathcal{D}_\Omega^\alpha(x) = \frac{\Gamma((n+\alpha)/2)}{2\pi^{(n-1)/2}\Gamma((1+\alpha)/2)}\int_{\Sigma_{n-1}}\Omega(\sigma)|x\cdot\sigma|^\alpha\,d\sigma, \tag{4.13}$$

$$\mathcal{D}_\Omega^\alpha(x) = -i\frac{\Gamma((n+\alpha)/2)}{2\pi^{(n-1)/2}\Gamma((1+\alpha)/2)}\int_{\Sigma_{n-1}}\Omega(\sigma)|x\cdot\sigma|^\alpha\operatorname{sgn}(x\cdot\sigma)\,d\sigma \tag{4.14}$$

for HSI's of neutral, even, and odd types, respectively.

COROLLARY 1. *HSI's do not depend on the order l for the choice* (4.7)–(4.9) *of the normalization constant $d_{n,l}(\alpha)$.*

COROLLARY 2. *The HSI's of neutral and even types coincide in the case of an even characteristic $\Omega(t') = \Omega(-t')$.*

It can also be concluded from (4.12)–(4.14) that for the integers $\alpha = 1,2,3,\ldots$ the HSI $\mathbf{D}_\Omega^\alpha f$ is a homogeneous differential operator of order α for a suitable choice of the type of HSI. Namely

COROLLARY 3. *For $\alpha = 2,4,6,\ldots$ the symbol of an HSI of neutral and even type, and for $\alpha = 1,3,5,\ldots$ the symbol of an HSI of odd type are polynomials:*

$$D_\Omega^\alpha(x) = \frac{\pi\delta}{2\beta_n(\alpha)}\sum_{|j|=\alpha}\frac{\Omega_j}{j!}x^j, \tag{4.15}$$

(4) In the case of an even characteristic $\Omega(t')$ the formulas (4.12) and (4.13) coincide and are true for HSI's of neutral type even for $\alpha = 1,3,5,\ldots$ (cf. also the formula (2.34) for the symbols of potentials).

where $\delta = 1$ for neutral or even type and $\delta = -i$ for odd type, and the Ω_j are the spherical moments of the function $\Omega(\sigma)$:

$$\Omega_j = \int_{\Sigma_{n-1}} \sigma^j \Omega(\sigma)\, d\sigma. \tag{4.16}$$

4. The hypersingular integral as a convolution with f.p. $\Omega(x')|x|^{-n-\alpha}$. Is it possible to regard an HSI as a convolution with a generalized function of the form f.p.$(\Omega(x')/|x|^{n+\alpha})$? Theorem 9 below provides a positive answer, in general, to this question. By definition,

$$\text{f.p.}\frac{\Omega(x')}{|x|^{n+\alpha}} * f = \int_{|t|<1} \Omega(t')\frac{f(x-t) - P_t^{l-1}(x)}{|t|^{n+\alpha}}\,dt + \int_{|t|>1} \frac{\Omega(t')f(x-t)}{|t|^{n+\alpha}}\,dt$$
$$+ \sum_{|j|\le l-1} \frac{(-1)^{|j|}}{j!}(D^j f)(x)\,\text{f.p.}\int_{|t|<1}\frac{t^j\Omega(t')}{|t|^{n+\alpha}}\,dt,$$

where

$$P_t^{l-1}(x) = \sum_{|j|\le l-1} (-t)^j \frac{1}{j!}(D^j f)(x), \qquad l > \alpha.$$

It is not hard to see that

$$\text{f.p.}\int_{|t|<1} t^j |t|^{-n-\alpha}\Omega(t')\,dt = \begin{cases} \Omega_j/(|j|-\alpha), & |j| \ne \alpha, \\ 0, & |j| = \alpha, \end{cases}$$

where the Ω_j are the spherical moments (4.16). Therefore,

$$\text{f.p.}\frac{\Omega(x')}{|x|^{n+\alpha}} * f = \int_{R^n} \Omega(t')\frac{f(x-t) - \chi(t)P_t^{l-1}(x)}{|t|^{n+\alpha}}\,dt$$
$$+ \sum_{|j|\le l-1}{}' \frac{(-1)^{|j|}\Omega_j}{j!\,(|j|-\alpha)}(D^j f)(x),$$

where the prime on the summation sign means that the terms with index $|j| = \alpha$ are omitted when α is an integer, and $\chi(t)$ is the characteristic function of the ball $|t| < 1$.

THEOREM 9. *Suppose that $\Omega(\sigma) \in L_1(\Sigma_{n-1})$, and that $f(x) \in C^l(R^n)$ and is bounded. Then*

$$(\mathbf{D}_\Omega^\alpha f)(x) = -\frac{\sin(\alpha\pi/2)}{\beta_n(\alpha)}\,\text{f.p.}\frac{\Omega(x')}{|x|^{n+\alpha}} * f, \qquad \alpha \ne 2,4,6,\ldots; \tag{4.17}$$

$$(\mathbf{D}_\Omega^\alpha f)(x) = -\frac{\sin(\alpha\pi/2)}{\beta_n(\alpha)}\,\text{f.p.}\frac{\Omega(x') + \Omega(-x')}{2|x|^{n+\alpha}} * f, \quad \alpha = 2,4,6,\ldots; \tag{4.18}$$

$$(\mathbf{D}_\Omega^\alpha f)(x) = -\frac{\cos(\alpha\pi/2)}{\beta_n(\alpha)}\,\text{f.p.}\frac{\Omega(x') - \Omega(-x')}{2|x|^{n+\alpha}} * f, \quad \alpha \ne 1,3,5,\ldots, \tag{4.19}$$

for HSI's of neutral, even, and odd types, respectively. In the cases of the integers α *excluded in* (4.17)–(4.19) *the HSI* $\mathbf{D}_\Omega^\alpha f$ *is a homogeneous differential operator of order* α:

$$(\mathbf{D}_\Omega^\alpha f)(x) = \frac{(-1)^{[\alpha/2]}\pi}{2\beta_n(\alpha)} \sum_{|j|=\alpha} \frac{\Omega_j}{j!}(D^j f)(x) \tag{4.20}$$

($\alpha = 2, 4, 6, \ldots$ *for neutral and even types and* $\alpha = 1, 3, 5, \ldots$ *for odd type*).

It is clear from Theorem 9 that the symbol $D_\Omega^\alpha(x)$ of the HSI $\mathbf{D}_\Omega^\alpha f$ can be regarded as the Fourier transform of the generalized function const f.p. $(\Omega(x')/|x|^{n+\alpha})$ or, perhaps, of its even or odd component. Note, however, that this function does not convolute the Schwartz space S, for example, into itself, and, therefore, we cannot (as also in §2.4) justify the transition to Fourier transforms by a simple reference to the Gel'fand-Shilov theorem ([5], Chapter III, §3.7, Theorem 1). The function $\Omega(x')/|x|^{n+\alpha}$ convolutes the Lizorkin space $\Psi = \hat{\Phi}$ into itself if $\Omega(x') \in C^\infty(\Sigma_{n-1})$. It is, however, possible to carry out the proof for $\Omega(x') \in L_1(\Sigma_{n-1})$. Regarding $\Omega(x')/|x|^{n+\alpha}$ as an element of the class Ψ', we may omit the f.p. symbol.

THEOREM 10. *Let* $\Omega(\sigma) \in L_1(\Sigma_{n-1})$. *Then*

$$D_\Omega^\alpha(x) = -\frac{\sin(\alpha\pi/2)}{\beta_n(\alpha)} F\left[\frac{\Omega(x')}{|x|^{n+\alpha}}\right], \tag{4.21}$$

$$D_\Omega^\alpha(x) = -\frac{\sin(\alpha\pi/2)}{\beta_n(\alpha)} F\left[\frac{\Omega(x') + \Omega(-x')}{2|x|^{n+\alpha}}\right], \tag{4.22}$$

$$D_\Omega^\alpha(x) = -\frac{\cos(\alpha\pi/2)}{\beta_n(\alpha)} F\left[\frac{\Omega(x') - \Omega(-x')}{2|x|^{n+\alpha}}\right] \tag{4.23}$$

for HSI's of neutral, even, and odd types, respectively; the Fourier transforms are understood in the sense of Φ'*-distributions.*(5)

5. Estimation of hypersingular integrals of smooth decreasing functions.

THEOREM 11. *If* $\Omega(\sigma)$ *is bounded, and* $f(x) \in C^l(R^n)$ *and*

$$|f(x)| \leq \frac{c}{(1+|x|)^{N_1}}, \qquad |(D^j f)(x)| \leq \frac{c}{(1+|x|)^{N_2}} \quad \text{for } |j| = l,$$

where $N_1 > \alpha$ *and* $N_2 > n$, *then*

$$|(\mathbf{D}_\Omega^\alpha f)(x)| \leq c(1+|x|)^{-\min(\alpha+N_1, N_2, n+\alpha)}.$$

6. On smoothness of the symbols of hypersingular integrals. Comparing the symbol (4.12)–(4.14) of a hypersingular integral with the formula (2.34)–(2.34′) for the symbol of a potential, we get the following result from Theorem 3.

ASSERTION. *The symbol* $D_\Omega^\alpha(x)$, $x \neq 0$, *of a hypersingular integral has order of smoothness greater by* $\alpha + 1$ *than that of the characteristic* $\Omega(\sigma)$:

$$\Omega(\sigma) \in C^\lambda(\Sigma_{n-1}) \Rightarrow D_\Omega^\alpha(x') \in C^{\lambda+\alpha+1}(\Sigma_{n-1}),$$

(5) Here we do not make separate provision for the cases of integral values of α, as in Theorem 9. These cases are contained in (4.21)–(4.23). The assertion $D_\Omega^\alpha(x) \equiv 0$ then obtained corresponds to the fact that the polynomials are indistinguishable from zero in the framework of the Φ' distributions (see [9]).

where $\lambda \geqslant 0$, $\alpha > 0$, *and* $\alpha \neq 1, 2, 3, \ldots$ *and* $\lambda - \alpha \neq 0, 1, 2, \ldots$. *But if* α *or* $\lambda - \alpha$ *is an integer, then*

$$D_\Omega^\alpha(x') \in C_*^{\lambda+\alpha+1}(\Sigma_{n-1}).$$

Indeed, in the case of a hypersingular integral of, for example, neutral type we have from (4.12) after differentiating $[\alpha] + 1$ times that

$$D^m D_\Omega^\alpha(x) = \text{const} \int_{\Sigma_{n-1}} \frac{\sigma^m \Omega(\sigma)\, d\sigma}{(-ix \cdot \sigma)^{1-(\alpha-[\alpha])}}$$

for a multi-index m of length $|m| = [\alpha] + 1$. It then remains to apply Theorem 3. An analysis of the proof of Theorem 3 shows that it is true also for integrals with singularities $1/|x \cdot \sigma|^\alpha$ and $\operatorname{sgn}(x \cdot \sigma)/|x \cdot \sigma|^\alpha$ instead of $1/(-ix \cdot \sigma)^\alpha$. Therefore, it can be applied also in the case of hypersingular integrals of even and odd types.

7. Hypersingular integrals with harmonic characteristic. In the case where $\Omega(t') = Y_m(t')$ is a spherical harmonic we have

THEOREM 12. *The HSI is annihilated,*

$$\mathbf{D}_{Y_m}^\alpha f = \frac{1}{d_{n,l}(\alpha)} \int_{R^n} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}} \mathbf{Y}_m\left(\frac{t}{|t|}\right) dt \equiv 0$$

in the following cases: a) *an integral of neutral type and either an even integer* α *less than* m *or* $\alpha = m + 1$, $m + 2, \ldots$; b) *an integral of even type and either odd* m *or* $\alpha = m - 2$, $m - 4, \ldots$; c) *an integral of odd type and either even* m *or* $\alpha = m - 2$, $m - 4, \ldots$.

In the remaining cases we have

$$\begin{aligned} \mathbf{D}_{Y_m}^\alpha f &= \lambda Y_m(D)\mathbf{D}^{\alpha-m} f \quad \text{for } \alpha \geqslant m, \\ \mathbf{D}_{Y_m}^\alpha f &= \lambda Y_m(D) K^{m-\alpha} f \quad \text{for } \alpha \leqslant m, \end{aligned} \tag{4.24}$$

where $f \in S(R^n)$, $K^{m-\alpha}$ is a Riesz potential of order $m - \alpha$, $\mathbf{D}^{\alpha-m} = \mathbf{D}_\Omega^{\alpha-m}|_{\Omega\equiv 1}$ is the Riesz differentiation operator, and

$$\lambda = E \frac{\Gamma((n+\alpha)/2)\Gamma(1+\alpha/2)}{\Gamma((n+\alpha+m)/2)\Gamma(1+(\alpha-m)/2)},$$

where

$$E = (-1)^m \cos\frac{\alpha+m}{2}\pi \Big/ \cos\frac{\alpha\pi}{2}, \quad E = (-1)^{m/2}, \quad E = (-1)^{(m+1)/2}$$

for the neutral, even, and odd types of hypersingular integral, respectively.

8. Representation of homogeneous differential operators by hypersingular integrals. First of all we mention a result that follows from the "reduction formula" (4.24).

COROLLARY. *The differential operators*

$$P_\alpha(D) = \Delta^k Y_m(D), \qquad \alpha = 2k + m, \qquad k = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots,$$

where Δ *is the Laplacian and* Y_m *is a homogeneous harmonic polynomial of order* m, *can be represented in the form of a hypersingular integral*

$$\Delta^k Y_m(D) f = \frac{\mu}{d_{n,l}(\alpha)} \int_{R^n} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}} Y_m\left(\frac{t}{|t|}\right) dt, \tag{4.25}$$

where

$$\mu = (-1)^{[m/2]} k!\Gamma\left(\frac{n}{2} + m + k\right) \Big/ \Gamma\left(\frac{n+m}{2} + k\right)\Gamma\left(\frac{m}{2} + k + 1\right),$$

and the HSI is of neutral or even type for even m and of odd type for odd m.

Theorem 9 asserted earlier (see (4.20)) that differential operators are contained among the hypersingular integrals. The above corollary suggests that the converse statement is valid: all the homogeneous differential operators are contained among the HSI's. This is indeed so. Namely,

THEOREM 13. *Suppose that $\alpha = 1, 2, 3, \ldots$ and $P_\alpha(D)$ is a homogeneous differential operator of order α. There exists a homogeneous polynomial $\Omega_\alpha(x)$ of order α such that*

$$P_\alpha(D)f = \frac{1}{d_{n,l}(\alpha)} \int_{R^n} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}} \Omega_\alpha\left(\frac{t}{|t|}\right) dt, \tag{4.26}$$

where the HSI on the right-hand side is of neutral or even type for $\alpha = 2, 4, 6, \ldots$ and of odd type for $\alpha = 1, 3, 5, \ldots$. The characteristic $\Omega_\alpha(t')$ of the HSI can be computed from the given polynomial $P_\alpha(x)$ by the formula

$$\Omega_\alpha(t') = \int_{\Sigma_{n-1}} P_\alpha(\sigma)\mathcal{K}(t' \cdot \sigma)\, d\sigma, \tag{4.27}$$

where

$$\mathcal{K}(y) = \frac{(-1)^{[\alpha/2]}\Gamma(n/2)}{\Gamma(1 + \alpha/2)\Gamma((n+\alpha)/2)} \times \sum_{k=0}^{[\alpha/2]} (-1)^k k!\Gamma\left(\frac{n}{2} + \alpha - k\right) d(\alpha - 2k) H_{\alpha-2k}(y),$$

and the $H_{\alpha-2k}$ are the Gegenbauer-Tchebycheff polynomials (1.12). *In particular, for $\alpha = 2$*

$$\Omega_\alpha(t') = \left[-\frac{n+2}{2} P_2(t') + \frac{1}{2} \operatorname{tr} P_2.\right] \tag{4.28}$$

§5. Inversion of the potentials $K_\theta^\alpha\varphi$ by hypersingular integrals

Our goal is to construct an operator inverse to K_θ^α in the elliptic case (3.1). For a formal solution of the equation $K_\theta^\alpha\varphi = f$ it is necessary, in view of (2.89), to form the convolution

$$\varphi = (1/\mathcal{K}_\theta^\alpha(x))^\sim * f, \tag{5.1}$$

where $(1/\mathcal{K}_\theta^\alpha(x))^\sim$ is the Fourier transform of $1/\mathcal{K}_\theta^\alpha(x)$. A justification of the operations in (5.1) in the framework of generalized functions encounters the difficulties mentioned earlier involving the fact that $(1/\mathcal{K}_\theta^\alpha)^\wedge$ does not convolute "reasonable" classes of generalized functions into themselves. After studying the properties of the function $(1/\mathcal{K}_\theta^\alpha)^\sim$ (generalized, in general, and ordinary if the order of smoothness of $\theta(x')$ is high enough) and using Theorem 8, we shall find when the inversion of a potential is possible in the form of an HSI. We effectively construct this integral, which is a candidate for the inverse operator, and show that it really does invert the potential. In the present section this will be done in the framework of C^∞-functions when the density φ of the potential is in the Lizorkin class $\Phi(R^n)$, and then in §6 it will be done in the framework of the L_p-spaces with the corresponding (in the L_p-norm) understanding of convergence for the HSI. The key role in the proof for L_p-densities is played by the property of "annihilation of the kernel of a potential by the HSI associated with it."

The inversion of a potential by an HSI turns out to be certainly possible when α is not an integer and in a number of cases for α an integer (for example, when a characteristic $\theta(\sigma)$ is even for $\alpha = 1, 3, 5, \ldots$ and odd for $\alpha = 2, 4, 6, \ldots$ (see the table below)). We also dwell briefly on the cases where inversion in the form (4.3) is not possible. In these cases the inverse operator will be: either 1) the sum of an HS operator and a differential operator (equal to a hypersingular integral of "mixed type" (see (5.30)); or 2) the composition of an HS operator and a singular operator.

1. Structure of the Fourier transform of the reciprocal of the symbol of a potential.

LEMMA 10. *Suppose that* $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, *where*

$$\lambda > 2\alpha + 2n - 1, \tag{5.2}$$

and that the ellipticity condition (3.1) *holds. Then*

$$\left(F^{-1}\frac{1}{\mathcal{K}_\theta^\alpha}\right)(x) = \begin{cases} f.p.\, \dfrac{\omega(x')}{|x|^{n+\alpha}}, & \alpha \text{ not an integer}, \\ f.p.\, \dfrac{\omega(x')}{|x|^{n+\alpha}} + P_\alpha(D)\delta(x), & \alpha = 1, 2, 3, \ldots, \end{cases} \tag{5.3}$$

where the Fourier transform is understood in the sense of the S' *distributions, and*

$$\omega(x') = \frac{2^\alpha}{\pi^{n/2}} \sum_{k,\mu} (-i)^k a_{k\mu} \frac{\Gamma((n+\alpha+k)/2)}{\Gamma((k-\alpha)/2)} Y_{k\mu}^{(x')}, \qquad a_{k\mu} = \int_{\Sigma_{n-1}} \frac{Y_{k\mu}(\sigma)}{\mathcal{K}_\theta^\alpha(\sigma)} d\sigma; \tag{5.4}$$

moreover, $\omega(x') \in C(\Sigma_{n-1})$, *and* $P_\alpha(D)$ *is a homogeneous differential operator of order* α:

$$P_\alpha(D) = i^\alpha \sum_{k=0}^{[\alpha/2]} \sum_{\mu=0}^{d(k)} a_{\alpha-2k,\mu} Y_{\alpha-2k,\mu}(D)\Delta^k. \tag{5.5}$$

PROOF. The expansion

$$\frac{1}{\mathcal{K}_\theta^\alpha(x')} = \sum_{k,\mu} a_{k\mu} Y_{k\mu}(x') \tag{5.6}$$

converges, by (3.1), because the symbol $\mathcal{K}_\theta^\alpha(x')$ is sufficiently smooth (this follows from (2.86)). We have

$$F^{-1}\left(\frac{1}{\mathcal{K}_\theta^\alpha}\right) = F^{-1}\left[|x|^\alpha \sum_{k,\mu} a_{k\mu} Y_{k\mu}(x')\right] = \sum_{k,\mu} a_{k\mu} F^{-1}\left[|x|^\alpha Y_{k\mu}(x')\right]$$

(the proof that the Fourier transformation is applicable to a series in the sense of S'-distributions is not difficult).

Applying (1.52), we obtain (5.3). Let us determine the nature of convergence of the series (5.4). From the asymptotic behavior of $\Gamma(z)$ at infinity (see [6], 8.327) it follows that

$$\frac{\Gamma(z+a)}{\Gamma(z-b)} \underset{z\to\infty}{\sim} z^{a+b}. \tag{5.7}$$

Therefore, (5.4) is majorized by the series

$$\sum_{k,\mu} |a_{k\mu}| k^{n/2+\alpha} | Y_{k\mu}(x')| \leq \sum_{k,\mu} |a_{k\mu}| k^{n+\alpha-1}.$$

Since the function (5.6) belongs to $C^{\lambda-\alpha+1}(\Sigma_{n-1})$ by (2.86), the majorizing series converges on the basis of Lemma 1 if $n+\alpha-1 < 2[(\lambda-a+1)/2]-n+1$, and for this it suffices that $\lambda > 2n+2\alpha-1$.

REMARK 16. $P_\alpha(D) \equiv 0$ in the case $\alpha = 2,4,6,\dots$ for an odd characteristic $\theta(t')$ and in the case $\alpha = 1,3,5,\dots$ for an even characteristic.

2. Representation of $(1/\mathcal{K}_\theta^\alpha)^{\check{}}$ by an f.p.-integral over the sphere. So far we have been able to determine from the symbol $\mathcal{K}_\theta^\alpha(x')$ the function $\omega(x')$ in (5.3) in the form of the series (5.4), assuming that we know the expansion of $1/\mathcal{K}_\theta^\alpha(x')$ in spherical harmonics. Is it possible to bypass this expansion and express $\omega(x')$ explicitly in terms of $1/\mathcal{K}_\theta^\alpha(x')$? The next theorem answers this question positively in terms of the f.p.-constructions on the sphere studied in §2.2.

THEOREM 14. *Suppose that* $\theta(\sigma) \in C^\lambda(\Sigma_{n-1})$, $\lambda > n+2\alpha-2$, *and that* (3.1) *holds. Then*

$$\omega(x') = \frac{\Gamma(n+\alpha)}{(2\pi)^n} f.p. \int_{\Sigma_{n-1}} \frac{d\sigma}{\mathcal{K}_\theta^\alpha(\sigma)(ix'\cdot\sigma)^{n+\alpha}} \tag{5.8}$$

for $\alpha \neq 1,2,3,\dots$. *But if* $\alpha = 1,2,3,\dots$, *then the following expression is added to the right-hand side of* (5.8) (*as in* (2.34')):

$$\frac{\pi i^{n+\alpha-1}}{(2\pi)^n} \frac{\partial^{n+\alpha-1}\tilde{M}_{1/\mathcal{K}_\theta^\alpha}(x',0)}{\partial y^{n+\alpha-1}}.$$

PROOF. First of all, we note that $(1/\mathcal{K}_\theta^\alpha(\sigma)) \in C^{\lambda-\alpha+1}(\Sigma_{n-1})$, by the assertion (2.86), and, therefore, the right-hand side of (5.8) exists for $\lambda > n+2\alpha-2$, by Theorem 2 (and even belongs to $C^{\lambda-n-2\alpha+2}(\Sigma_{n-1})$, by Theorem 3). It is necessary to prove the equality

$$\int_{R^n} \frac{\varphi(x)\,dx}{\mathcal{K}_\theta^\alpha(x)} = \frac{\Gamma(n+\alpha)}{(2\pi)^n} \int_{R^n} \frac{\hat{\varphi}(x)}{|x|^{n+\alpha}} \left(\text{f.p.} \int_{\Sigma_{n-1}} \frac{d\sigma}{\mathcal{K}_\theta^\alpha(\sigma)(ix'\cdot\sigma)^{n+\alpha}} \right) dx \tag{5.9}$$

for $\varphi(x) \in \Phi(R^n)$. The left-hand side reduces to the form

$$\begin{aligned} \int_{R^n} \frac{|x|^\alpha \varphi(x)}{\mathcal{K}_\theta^\alpha(x')} dx &= \frac{1}{(2\pi)^n} \int_0^\infty \rho^{n+\alpha-1}\,d\rho \int_{\Sigma_{n-1}} \frac{d\sigma}{\mathcal{K}_\theta^\alpha(\sigma)} \int_{R^n} e^{-i\rho\sigma\cdot t} \hat{\varphi}(t)\,dt \\ &= \frac{1}{(2\pi)^n} \lim_{N\to\infty} \int_{R^n} \frac{\hat{\varphi}(t)\,dt}{|t|^{n+\alpha}} \int_{\Sigma_{n-1}} \frac{d\sigma}{\mathcal{K}_\theta^\alpha(\sigma)} \int_0^{N|t|} e^{-i\rho\sigma(t/|t|)} \rho^{n+\alpha-1}\,d\rho, \end{aligned} \tag{5.10}$$

and it remains to refer to Remark 13; it is not difficult to justify taking the limit under the sign of the integral over R^n in (5.10), since $\hat{\varphi}(t)|t|^{-\alpha-n} \in S(R^n)$ for $\varphi(t) \in \Phi(R^n)$.

COROLLARY. *An explicit expression for the function $\omega(x')$ in terms of the characteristic $\theta(\sigma)$ follows from* (5.8) *and* (2.34) (*for* $\alpha \neq 1, 2, 3, \ldots$):

$$\omega(x') = \frac{\Gamma(n+\alpha)}{(2\pi)^n\Gamma(\alpha)} f.p. \int_{\Sigma_{n-1}} \frac{d\sigma}{(ix'\cdot\sigma)^{n+\alpha} f.p. \int_{\Sigma_{n-1}} (-i\sigma\cdot\tau)^{-\alpha}\theta(\tau)\,d\tau}. \tag{5.11}$$

3. The associated characteristics. By virtue of the formal equality (5.1) and formula (5.3), we expect that the inverse operator $(K_\theta^\alpha)^{-1}$ will be given by the equality

$$\varphi(x) = \text{f.p.} \frac{\omega(x')}{|x|^{n+\alpha}} * f \qquad \text{if } \alpha \text{ is not an integer,} \tag{5.12}$$

$$\varphi(x) = \text{f.p.} \frac{\omega(x')}{|x|^{n+\alpha}} * f + P_\alpha(D)f \quad \text{if } \alpha = 1, 2, 3, \ldots, \tag{5.13}$$

where $P_\alpha(D)$ is the differential operator (5.5). It can be reduced to the form

$$P_\alpha(D) = i^\alpha \frac{\Gamma(n/2)}{2\pi^{n/2}} \int_{\Sigma_{n-1}} \frac{Q_\alpha(\sigma; D)}{\mathcal{K}_\theta^\alpha(\sigma)} d\sigma, \tag{5.14}$$

where

$$Q_\alpha(\sigma; D) = \sum_{k=0}^{[\alpha/2]} d(\alpha - 2k)\Delta^k H_{\alpha-2k}(\sigma\cdot D),$$

$\sigma\cdot D = \sigma_1(\partial/\partial x_1) + \cdots + \sigma_n(\partial/\partial x_n)$, and the H_m are the polynomials (1.12). For this it suffices to substitute $a_{k\mu}$ from (5.4) into (5.5) and use the addition formula for spherical harmonics. For $n \geqslant 3$ it is possible further to write

$$Q_\alpha(\sigma; D) = \sum_{k=0}^{[\alpha/2]} \frac{n - 2 + 2\alpha - 4k}{n-2} \Delta^k C_{\alpha-2k}^{(n-2)/2}(\sigma\cdot D),$$

while in the planar case $n = 2$ (for α an integer it is necessary that $\alpha = 1$) $Q_\alpha(\sigma; D) = 2\sigma\cdot D$. Thus,

$$P_\alpha(D) = a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} \quad \text{for } n = 2, \alpha = 1, \tag{5.15}$$

where

$$a_1 = \frac{i}{\pi}\int_0^{2\pi} \frac{\cos\xi}{\mathcal{K}_\theta^\alpha(e^{i\xi})} d\xi, \qquad a_2 = \frac{i}{\pi}\int_0^{2\pi} \frac{\sin\xi}{\mathcal{K}_\theta^\alpha(e^{i\xi})} d\xi.$$

The arguments which led us to the form (5.12)–(5.13) for the inverse operator should so far be regarded as heuristic, and we must still prove that the inverse operator that we shall construct, starting from (5.12)–(5.13), really is the inverse. We shall ascertain when the construction (5.12)–(5.13) can be realized in the form of an

HSI, and this construction will be carried out. The decisive role here belongs to Theorem 9, which enables us to pass to the HSI (excluding the "polynomial" cases)

$$\text{f.p.}\,\frac{\omega(x')}{|x|^{n+\alpha}} * f = \mathbf{D}_\Omega^\alpha f. \tag{5.16}$$

The characteristic $\Omega(x')$ in this case has, by Theorem 9, the form

$$\Omega(x') = -\frac{\beta_n(\alpha)}{\sin(\alpha\pi/2)}\omega(x'), \qquad \alpha \neq 2,4,6,\ldots, \tag{5.17}$$

when we want to use an HSI $D_\Omega^\alpha f$ of neutral or even type (here $\omega(x')$ must be even, according to (4.18), for even type). But in the case of an HSI of odd type

$$\Omega(x') = -\frac{\beta_n(\alpha)}{\cos(\alpha\pi/2)}\omega(x'), \qquad \alpha \neq 1,3,5,\ldots \tag{5.17'}$$

(and then $\omega(x')$ must be odd, according to (4.19)).

DEFINITION 7. The characteristic $\Omega(x')$ constructed from the characteristic $\theta(x')$ according to the rule (5.17)–(5.17′), where $\omega(x')$ is the function (5.4), (5.8), is called the *characteristic associated with* $\theta(x')$. Also, the HSI $\mathbf{D}_\Omega^\alpha f$ with this characteristic $\Omega(x')$ is said to be *associated with the potential* $K_\theta^\alpha\varphi$.

For sufficiently smooth characteristics $\theta(x') \in C^\lambda(\Sigma_{n-1})$ the associated characteristic $\Omega(x')$ exists as an ordinary function (continuous for $\lambda \geqslant 2\alpha + 2n - 1$), by Lemma 10.

REMARK 17. If the characteristic $\theta(x')$ of a potential is even (odd), then so is the associated characteristic $\Omega(x')$ of the HSI; see (5.8) and (2.34)–(2.34′).

REMARK 18. It can be shown that if the characteristic $\theta(t')$ depends only on a single variable, $\theta = \theta(t_j/|t|)$, then the characteristic $\Omega(t')$ associated with it also depends only on this variable. Moreover, $\Omega(t')$ can be constructed effectively in terms of an expansion in Gegenbauer-Tchebycheff polynomials.

4. Inversion of potentials of noninteger order α by hypersingular integrals. The next theorem is a consequence of the constructions in subsections 2 and 3.

THEOREM 15. *Suppose that the characteristic $\theta(x')$ of the potential $K_\theta^\alpha\varphi$, $\alpha \neq 1,2,3,\ldots$, satisfies the smoothness assumption* (5.2) *and the ellipticity condition* (3.1). *Then the HSI of neutral type with characteristic $\Omega(x')$ associated with $\theta(x')$ inverts the potential K_θ^α*:

$$\mathbf{D}_\Omega^\alpha K_\theta^\alpha\varphi \equiv \varphi, \qquad \varphi \in \Phi(R^n). \tag{5.18}$$

If $\theta(x')$ is even (odd), then $\mathbf{D}_\Omega^\alpha$ can also be taken of even (odd) type.

PROOF. Since "nice" functions $\varphi \in \Phi(R^n)$ are being considered, the equality

$$\mathfrak{D}_\Omega^\alpha(x)\mathcal{K}_\theta^\alpha(x) \equiv 1 \tag{5.19}$$

for the symbols can be proved instead of (5.18). [The passage from (5.18) to the Fourier transforms is easy to justify, although $\mathfrak{D}_\Omega^\alpha(x)$ and $\mathcal{K}_\theta^\alpha(x)$ are not, in general, multipliers. To do this note that if $\varphi \in \Phi$, then $K_\theta^\alpha\varphi$ is infinitely differentiable and decreases sufficiently rapidly at infinity, by Corollary 2 to Theorem 4 (see also §4.3 and §2.4).]

1. Suppose that $\mathbf{D}^{\alpha}_{\Omega} f$ is of neutral type. By (4.12), it must be checked that

$$\int_{\Sigma_{n-1}} \Omega(\sigma)(-ix\cdot\sigma)^{\alpha}\,d\sigma = \frac{2\pi^{(n-1)/2}\Gamma((1+\alpha)/2)\cos(\alpha\pi/2)}{\Gamma((n+\alpha)/2)}\frac{1}{\mathcal{K}^{\alpha}_{\theta}(x)}. \qquad (5.20)$$

Both sides of the equality here are homogeneous of order α; therefore, it suffices to prove it on the sphere Σ_{n-1}. Substituting $\Omega(\sigma)$ from (5.17), we make use of the expansion (5.4), (5.6). Equality (5.20) can be reduced to

$$\int_{\Sigma_{n-1}} \sum_{k,\mu} (-i)^k a_{k\mu}\frac{\Gamma((n+\alpha+k)/2)}{\Gamma((k-\alpha)/2)} Y_{k\mu}(\sigma)(-ix'\cdot\sigma)^{\alpha}\,d\sigma$$
$$= -\sin\alpha\pi\pi^{(n-3)/2}\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(1+\frac{a}{2}\right)\sum_{k,\mu} a_{k\mu}Y_{k\mu}(x').$$

It is not particularly difficult to justify the termwise integration of the series on the left-hand side here. Then applying (1.16), we come to a comparison of the coefficients of the $Y_{k\mu}(x')$, which really turn out to be equal.

2. Suppose that $\theta(x')$ is even. Then so is $\Omega(x')$ (see Remark 17). Therefore, (4.12) and (4.13) coincide, and the symbol $\mathfrak{D}^{\alpha}_{\Omega}(x)$ of an HSI of even type is given by the previous formula. Therefore, the preceding proof is preserved.

3. Suppose that $\theta(x')$ is odd. We use an HSI of odd type. The preceding proof is preserved in principle, but the form of the formulas changes somewhat. It is now necessary to apply (4.14) instead of (4.12) and (1.20) instead of (1.16).

We mention that the use of HSI's not of neutral type only for even or for odd characteristics $\theta(t')$ is due to the essence of the matter; see (4.5).

5. Inversion of potentials of even order $\alpha = 2, 4, 6, \ldots$ by hypersingular integrals. The passage from the proposed inversion (5.3) to the HS construction is no longer always possible. According to Theorem 8, for $\alpha = 2, 4, 6, \ldots$ we have that: A) an HSI of neutral or even type arises only if it is a differential operator; and B) an HSI of odd type can be used only for odd $\Omega(x')$ and, consequently, for odd $\theta(x')$.

A) *Inversion by hypersingular integrals of neutral or even type.* In this case the symbol $\mathfrak{D}^{\alpha}_{\Omega}(x)$ of an HSI is, by Corollary 3 to Theorem 8, a homogeneous polynomial of order α. Consequently, by (5.19), it is necessary that $\mathcal{K}^{\theta}_{\alpha}(x) = 1/P_{\alpha}(x)$, where $P_{\alpha}(x)$ is a homogeneous polynomial of order α. Hence,

$$\frac{\theta(x')}{|x|^{n-\alpha}} = \left(\frac{1}{P_{\alpha}}\right)^{\check{}}(x), \qquad (5.21)$$

and we arrive at the following assertion:

Inversion of a potential K^{α}_{θ} of even order $\alpha = 2, 4, 6, \ldots$ by an HSI $\mathbf{D}^{\alpha}_{\Omega}$ of neutral or even type is possible if and only if the characteristic $\theta(x')$ of the potential is the restriction to the unit sphere of the fundamental solution $(1/P_{\alpha})^{\check{}}$ of some elliptic homogeneous differential operator $P_{\alpha}(D)$ of order α.

In the case $\alpha = 2$ it is possible to get the following more interesting statement.

THEOREM 16. *Potentials of the form*

$$K^2_{\theta}\varphi \equiv \int_{R^n} \frac{\theta((x-t)/|x-t|)}{|x-t|^{n-2}}\varphi(t)\,dt = f(x), \qquad n \geqslant 3, \qquad (5.22)$$

with real characteristic $\theta(\sigma)$ have an HSI of neutral or even type as an inverse operator if and only if

$$\theta(\sigma) = \pm[P_2(\sigma)]^{(2-n)/2}, \tag{5.23}$$

where $P_2(x) = A(x, x)$ is a positive-definite quadratic form of n variables. Moreover,

$$\varphi(x) = \mp\frac{\Gamma((n-2)/2)}{4\pi^{n/2}}\sqrt{\det P_2}\,\tilde{P}_2(D)f$$

$$= \pm\frac{\Gamma((n-2)/2)}{8\pi(n/2)}\sqrt{\det P_2}\,\mathbf{D}^2_\Omega f, \qquad f \in S, \tag{5.24}$$

where $\Omega(t') = (n+2)\tilde{P}_2(t') - \operatorname{tr}\tilde{P}_2$, and $\tilde{P}_2(x)$ is the quadratic form conjugate to $P_2(x)$.

PROOF. By the preceding assertion (see (5.21)), the symbol $\mathcal{K}^\alpha_\theta(x)$ is the quantity $1/P_2(x)$ inverse to a quadratic form of n variables. Since $P_2(x)$ is even, the symbol $\mathcal{K}^\alpha_\theta(x)$ is then even. The symbol is real because the function $\theta(\sigma)$ is real. Consequently, the quadratic form $P_2(x)$ is real and of a definite sign. It is known ([4], Chapter IV, §2.2) that

$$\frac{1}{[P_2(x)]^{(n-2)/2}} = \frac{1}{2^{n-2}\pi^{n/2}\Gamma((n-2)/2)\sqrt{\det P_2}}\left(\frac{1}{\tilde{P}_2}\right)^{\wedge} \tag{5.25}$$

for a positive-definite form $P_2(x)$. Here $\tilde{P}_2$ is the quadratic form conjugate to P_2 (i.e., such that the matrices corresponding to them are mutual inverses).

We remark that if a quadratic form is positive definite, then the form conjugate to it is also positive definite. Indeed, by Sylvester's criterion, it is necessary to derive the positivity of the principal minors of the matrix for the form $\tilde{P}_2$ from the positivity of those of the matrix for P_2. This follows from the relation

$$A^{-1}\begin{Vmatrix} i_1 & i_2 & \cdots & i_p \\ i_1 & i_2 & \cdots & i_p \end{Vmatrix} = \frac{A\begin{Vmatrix} i'_1 & i'_2 & \cdots & i'_{n-p} \\ i'_1 & i'_2 & \cdots & i'_{n-p} \end{Vmatrix}}{A\begin{Vmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{Vmatrix}} > 0$$

between the principal minors of mutually inverse matrices ([2], Chapter I, §4, (33)).

But then it follows from (5.21) and (5.25) that the general form of the characteristics $\theta(x')$ for which $1/\mathcal{K}^\alpha_\theta(x)$ is a quadratic form of definite sign is actually given by (5.23). The inversion formula (5.24) also follows from (5.25), since (5.25) means that the symbol $\mathcal{D}^\alpha_\Omega(x)$ of the inverting operator is the polynomial

$$\frac{\Gamma((n-2)/2)}{4\pi^{n/2}}\sqrt{\det P_2}\,\tilde{P}_2(x).$$

The passage to the HSI in (5.24) is realized by (4.26) and (4.28).

B) *Inversion by hypersingular integrals of odd type.* Unlike in case A), where the HS construction for $\alpha = 2, 4, 6, \ldots$ works only for special characteristics $\theta(\sigma)$, an HS construction of odd type turns out to be admissible for any odd $\theta(\sigma)$.

THEOREM 17. *Suppose that the characteristic $\theta(x')$ of a potential $K_\theta^\alpha\varphi$ of even order $\alpha = 2, 4, 6, \dots$ satisfies conditions* (3.1) *and* (5.2) *and is odd. Then the HSI of odd type with the characteristic $\Omega(x')$ associated with $\theta(x')$ inverts the potential $K_\theta^\alpha\varphi$:*

$$\mathbf{D}_\Omega^\alpha K_\theta^\alpha \varphi \equiv \varphi, \qquad \varphi \in \Phi(R^n).$$

The proof is the same as that of Theorem 15 (see case 3 in the proof of Theorem 15; Remark 16 should also be taken into account).

6. Inversion of potentials of odd order $\alpha = 1, 3, 5, \dots$ by hypersingular integrals.

A) *Inversion by hypersingular integrals of neutral or even type.* The following result is symmetric to Theorem 17.

THEOREM 17′. *Suppose that $\theta(x')$ satisfies conditions* (3.1) *and* (5.2) *and is even. Then for $\alpha = 1, 3, 5, \dots$ the HSI of neutral type (with the necessary choice $l = \alpha$) or even type with the characteristic $\Omega(x')$ associated with $\theta(x')$ inverts the potential $K_\theta^\alpha\varphi$.*

B) *Inversion by hypersingular integrals of odd type.* By Corollary 3 to Theorem 8, the symbol $\mathfrak{D}_\Omega^\alpha(x)$ of the HSI is a homogeneous polynomial of order α. Since the symbol $\mathcal{K}_\theta^\alpha(x)$ of the potential takes finite values for $x \neq 0$, it follows from (5.19) that this polynomial must be elliptic. However, here $\alpha = 1, 3, 5, \dots$, and obviously there are no elliptic polynomials of odd order with real coefficients at all, and none with complex coefficients for $n \geqslant 3$ (see [35], Proposition 20.1). There remains the single case $n = 2$, and, consequently, $\alpha = 1$. A polynomial $P_1(x) = c_1x_1 + c_2x_2 = c \cdot x$, $c = (c_1, c_2) \in C^2$, with complex coefficients $c_1 = a_1 + ib_1$, $c_2 = a_2 + ib_2$ is elliptic if and only if

$$\det c = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0. \tag{5.26}$$

We have the formula

$$(1/(c \cdot x))\hat{} = \frac{2\pi i \operatorname{sgn}(\det c)}{d \cdot x}, \qquad \det c \neq 0, \tag{5.27}$$

where $d = (d_1, d_2)$, $d_1 = b_2 - b_1 i$, and $d_2 = -a_2 + ia_1$. Indeed, (1.52) (for $n = 2$ and $\alpha = 1$) gives us

$$F\left(\frac{1}{x_1 + ix_2}\right) = F\left(\frac{x_1 - ix_2}{|x|^2}\right) = 2\pi i \frac{x_1 - ix_2}{|x|^2} = \frac{2\pi i}{x_1 + ix_2}, \tag{5.28}$$

which yields (5.27) in the case $P_1(x) = x_1 + ix_2$. The general formula (5.27) follows from (5.28) on the basis of the well-known ([43], p. 108) formula

$$(f(A\,\cdot))\hat{}(x) = |\det A|^{-1}\hat{f}(A^{*-1}x),$$

where A is a linear transformation in R^n. Formula (5.27) leads to the following theorem.

THEOREM 16. *Two-dimensional ($n = 2$) potentials* (2.1) *of integer order $\alpha = 1$ admit inversion in the form of an HSI of odd type if and only if their characteristics have the form $\theta(x') = 1/(c \cdot x')$, where $c = (c_1, c_2) \in C^2$ and* (5.26) *holds, i.e., if and only if*

$$K_\theta^1\varphi \equiv \iint_{R^2} \frac{\varphi(t_1, t_2)\, dt_1\, dt_2}{c_1(x_1 - t_1) + c_2(x_2 - t_2)} = f(x_1, x_2).$$

Moreover,

$$\varphi(x_1, x_2) = \frac{\operatorname{sgn}(\det c)}{2\pi}\left[(b_2 - b_1 i)\frac{\partial f}{\partial x_1} + (-a_2 + ia_1)\frac{\partial f}{\partial x_2}\right],$$

or, what is the same, $\varphi = \mathbf{D}^1_\Omega f$, *where* $\mathbf{D}^1_\Omega f$ *is the HSI of odd type with characteristic*

$$\Omega(x') = \frac{2\operatorname{sgn}(\det c)}{\pi^2}\left[(b_2 - b_1 i)\frac{x_1}{|x|} + (-a_2 + ia_1)\frac{x_2}{|x|}\right].$$

For convenience the results obtained on inversion of potentials are summarized in a table. In it we assume that the characteristic $\theta(x')$ of the potential satisfies the smoothness condition (5.2) and the ellipticity condition (3.1).

Table of invertibility of potentials by hypersingular integrals

Type of hypersingular integral	α	Condition on the characteristic $\theta(t')$ of the potential under which inversion by an HSI is possible	Characteristic of the inverting HS integral
neutral	$\alpha \neq 1, 2, 3, 4\dots$	no reservations	$\Omega(x') = -\frac{\beta_n(a)}{\sin(a\pi/2)}\omega(x')$
	$\alpha = 2, 4, 6, \dots$	$\theta(t')$ is the restriction to the sphere of the fundamental solution of a homogeneous elliptic differential operator of order α. In particular, for $\alpha = 2$ $\theta(t') = \pm[P_2(t')]^{(2-n)/2}$	$\Omega(x')$ is a polynomial. In particular, for $\alpha = 2$ $\Omega(x') = \pm[(n+2)\tilde{P}_2(x') - \operatorname{tr}\tilde{P}_2]\frac{1}{8}\pi^{-2} \times\Gamma\left(\frac{n-2}{2}\right)\sqrt{\det P_2}$
	$\alpha = 1, 3, 5, \dots$	$\theta(t')$ even	$\Omega(x') = -\frac{\beta_n(a)}{\sin(a\pi/2)}\omega(x')$
even	$\alpha \neq 2, 4, 6, \dots$	$\theta(t')$ even	$\Omega(x') = -\frac{\beta_n(a)}{\sin(a\pi/2)}\omega(x')$
	$\alpha = 2, 4, 6, \dots$	$\theta(t')$ is the restriction to the sphere of the fundamental solution of a homogeneous elliptic differential operator of order α. In particular, for $\alpha = 2$ $\theta(t') = \pm[P_2(t')]^{(2-n)/2}$	$\Omega(x')$ is a polynomial. In particular, for $a = 2$ $\Omega(x') = \frac{\Gamma((n-2)/2)}{8\pi^2}\sqrt{\det P_2} \times[(n+2)\tilde{P}_2(x') - \operatorname{tr}\tilde{P}_2]$
odd	$\alpha \neq 1, 3, 5\dots$	$\theta(t')$ odd	$\Omega(x') = -\frac{\beta_n(a)}{\cos(a\pi/2)}\omega(x')$
	$\alpha = 1, 3, 5\dots$	possible only for $n = 2$, $\alpha = 1$, and $\theta(t') = (c_1x_1 + c_2x_2)^{-1}$, $\operatorname{Im} c_1\bar{c}_2 \neq 0$	$\Omega(x') = \frac{\operatorname{sgn}(\operatorname{Im} c_1\bar{c})}{\pi^2} \times\left[\{\overline{c_1 - ic_2} - (c_1 + ic_2)\}\frac{x_1}{|x|} + i\{c_1 + ic_2 + (\overline{c_1 - ic_2})\}\frac{x_2}{|x|}\right]$

7. Inversion of potentials of integer order in the general case. In subsections 4–6 we established (see the table) that inversion of the potentials K_θ^α by HSI's is always possible for α not an integer, while for $\alpha = 1, 2, 3, \ldots$ it is possible without restriction in the case of $\alpha = 2, 4, 6, \ldots$ and odd $\theta(x')$ and in the case of $\alpha = 1, 3, 5, \ldots$ and even $\theta(x')$. In the remaining cases the characteristic $\theta(x')$ is restricted to be of a special form (see Theorems 16 and 16′).

Suppose now that $\theta(\sigma)$ is an arbitrary characteristic satisfying conditions (5.2) and (3.1). On "nice" functions $f \in S(R^n)$ the inverse operator can always be constructed on the basis of (5.3) in the form

$$(K_\theta^\alpha)^{-1} f = P_\alpha(D) f + \text{f.p.} \frac{\omega(x')}{|x|^{n+\alpha}} * f, \qquad \alpha = 1, 2, 3, \ldots, \tag{5.29}$$

however, it is not always possible to pass to an HSI. We single out the case (which generalizes somewhat the case of evenness or oddness) where the inverse operator can be formed from HSI's of different types.

Let $(\Delta_t^l f)'(x)$ and $(\Delta_t^m f)''(x)$ be the noncentered and centered differences, respectively.

DEFINITION 8. An integral of the form

$$\int_{R^n} \frac{\Omega_1(t/|t|)(\Delta_t^l f)'(x) + \Omega_2(t/|t|)(\Delta_t^m f)''(x)}{|t|^{n+\alpha}} dt \tag{5.30}$$

will be called an HSI of *mixed type*.

DEFINITION 9. A function defined on the unit sphere is said to be *almost even* (*odd*) *of order* α if it is the sum of an even (odd) function and the restriction $P_{\alpha-1}(x/|x|)$ to the sphere of some polynomial $P_{\alpha-1}(x)$ of degree $\alpha - 1$.

1. Let $\alpha = 2, 4, 6, \ldots$. The second term in (5.29) leads to an HSI (of odd type) on the basis of Theorem 9 (see (4.19)) if the function $\omega(x')$ is odd. By (5.4), this is possible if $a_{k\mu} = 0$ for $k = \alpha + 2, \alpha + 4, \ldots$. For the function $1/\mathcal{K}_\theta^\alpha(x)$ this means almost oddness of order α.

2. Let $\alpha = 1, 3, 5, \ldots$. The same Theorem 9 enables us to represent the second term in the form of an HSI of neutral or even type. Here (4.18) and Remark 16 force the function $\omega(x')$ to be even. We conclude from (5.4) that evenness of $\omega(x')$ is almost evenness of order α of the function $1/\mathcal{K}_\theta^\alpha(x)$.

As for the first (differential) term in (5.29), Theorem 13 allows us to write it also as an HSI. It will be of neutral or even type for $\alpha = 2, 4, 6, \ldots$ and of odd type for $\alpha = 1, 3, 5, \ldots$: just the opposite of the second term in (5.29). We arrive at the following result.

THEOREM 18. *If* $1/\mathcal{K}_\theta^\alpha(x)$ *is almost odd* (*even*) *of order* α *for even* (*odd*) α, *then the operator inverse to the potential* $K_\theta^\alpha \varphi$ *can be constructed in the form of an HSI of mixed type* (5.30).

In the general case, when neither almost evenness nor almost oddness necessarily holds, we restrict ourselves to the assertion that the inverse operator can be obtained in the form

$$(K_\theta^\alpha)^{-1} = P_\alpha(D) + (\lambda I + N)\mathbf{D}^\alpha,$$

where $P_\alpha(D)$ is the differential operator (5.5), $\mathbf{D}^\alpha$ is the Riesz differentiation operator ($\mathbf{D}^\alpha = \mathbf{D}^\alpha_\Omega|_{\Omega\equiv 1}$),

$$\lambda = \begin{cases} 0, & \alpha = 2,4,6,\dots, \\ \dfrac{\Gamma(n/2)}{2\pi^{n/2}} \displaystyle\int_{\Sigma_{n-1}} \frac{d\sigma}{\mathcal{K}^\alpha_\theta(\sigma)}, & \alpha = 1,3,5,\dots, \end{cases}$$

and

$$N\varphi = \text{p.v.} \int_{R^n} \frac{\tau((x-t)/|x-t|)}{|x-t|^n} \varphi(t)\, dt$$

is the multi-dimensional Calderón-Zygmund operator with characteristic

$$\tau(x') = \frac{1}{\pi^{n/2}} \sum_{k,\mu} (-i)^k \frac{\Gamma((n+k)/2)}{\Gamma(k/2)} a_{k\mu} Y_{k\mu}(x'), \quad k \neq 0, \quad k \neq \alpha, \alpha-2, \dots,$$

which has zero mean value. Here the $a_{k\mu}$ are the coefficients in (5.4) and (5.6).

§6. Inversion of the potentials $K^\alpha_\theta\varphi$ by hypersingular integrals (extension; convergence in L_p)

Let us now invert the potential $f = K^\alpha_\theta\varphi$, by finding solutions $\varphi(x) \in L_p(R^n)$, $1 \leq p < n/\alpha$. The HS construction will no longer converge in the usual sense; it will be the limit in the L_p-norm of the corresponding truncated HSI's. Two central points precede this result: the annihilation of the kernel of a Riesz potential by the associated HSI (subsection 1), and the integral representation of the truncated HSI's (subsection 3).

Let $k^\alpha_\theta(x) = \theta(x')/|x|^{n-\alpha}$ be the kernel of the potential. The conditions (5.2) and (3.1) for $\theta(x')$ are assumed to hold. In this section the characteristic $\theta(x')$ is assumed to be arbitrary when α is not an integer, even for $\alpha = 2,4,6,\dots,$ and odd for $\alpha = 1,3,5,\dots$ [so that $P_\alpha(D) \equiv 0$ in (5.29) (see Remark 16), and it is always possible to invert by an HSI (see Theorems 15, 17, and 17′)].

Let

$$\mathbf{D}^\alpha_{\Omega,\varepsilon} f = \frac{1}{d_{n,l}(\alpha)} \int_{|t|>\varepsilon} \frac{(\Delta^l_t f)(x)}{|t|^{n+\alpha}} \Omega\left(\frac{t}{|t|}\right) dt \tag{6.1}$$

denote a truncated HSI.

1. The annihilation of the kernel of a potential by the associated hypersingular integral.

THEOREM 19. *The HSI with the characteristic $\Omega(x')$ associated with $\theta(x')$ annihilates the kernel of the potential*:

$$(\mathbf{D}^\alpha_\Omega k^\alpha_\theta)(x) = \frac{1}{d_{n,l}(\alpha)} \int_{R^n} \frac{(\Delta^l_t k^\alpha_\theta)(x)}{|t|^{n+\alpha}} \Omega(t')\, dt \equiv 0 \tag{6.2}$$

for all $x \in R^n \setminus \{0\}$. The interpretation of (6.2) *in the framework of generalized functions* (*over S*) *is*

$$\mathbf{D}^\alpha_\Omega k^\alpha_\theta = \delta, \tag{6.3}$$

where $\delta = \delta(x)$ is the delta function concentrated at the origin.

PROOF. First of all we mention that the integral in (6.2) converges for $x \neq 0$. Formally, (6.3) follows from the relation (5.19) between the symbol $\mathfrak{D}_\Omega^\alpha(x)$ and the symbol $\mathcal{K}_\theta^\alpha(x)$ in the case of the associated characteristics $\Omega(t')$ and $\theta(t')$. However, a direct justification of this approach encounters essential difficulties connected with the bad behavior of $k_\theta^\alpha(x)$ as $x \to 0$ ("bad" for the HSI D_Ω^α). And, of course, it is not possible here to use the Gel'fand-Shilov theorem on passing from the convolution $D_\Omega^\alpha k_\theta^\alpha$ to the product of the symbols.

Plan of the proof: 1) We introduce the functional

$$\langle G, \varphi \rangle = \lim_{\varepsilon \to 0} \int_{R^n} \overline{\varphi(x)}\,(\mathbf{D}_{\Omega,\varepsilon}^\alpha k_\theta^\alpha)(x)\,dx \tag{6.4}$$

and prove that it has the representation

$$\langle G, \varphi \rangle = \int_{R^n} k_\theta^\alpha(x)(\mathbf{D}_{\Omega^*}^\alpha \tilde{\varphi})(x)\,dx, \tag{6.5}$$

where $\Omega^*(x) = \Omega(-x)$, then show that this is a continuous functional in $S(R^n)$.

2) Using the representation (6.5), we show that $\hat{G} \equiv 1$, from which it will follow that $\langle G, \varphi \rangle \equiv \langle \delta, \varphi \rangle$.

3) On the basis of the definition (6.4) we derive the required equality (6.2) from 2).

1. Suppose for definiteness that the HSI D_Ω^α is of neutral type. We have

$$\begin{aligned}
\langle G, \varphi \rangle &= \frac{1}{d_{n,l}(\alpha)} \lim_{\varepsilon \to 0} \left\{ \int_{R^n} \overline{\varphi(x)}\,dx \int_{|t|>\varepsilon} \frac{k_\theta^\alpha(x)}{|t|^{n+\alpha}} \Omega(t')\,dt \right. \\
&\quad \left. + \sum_{\nu=1}^{l} (-1)^\nu C_l^\nu \nu^\alpha \int_{R^n} \overline{\varphi(x)}\,dx \times \int_{|x-t|>\nu\varepsilon} \frac{k_\theta^\alpha(t)}{|x-t|^{n+\alpha}} \Omega\left(\frac{x-t}{|x-t|}\right) dt \right\} \\
&= \frac{1}{d_{n,l}(\alpha)} \lim_{\varepsilon \to 0} \left\{ \int_{R^n} k_\theta^\alpha(x)\,dx \int_{|t|>\varepsilon} \frac{\overline{\varphi(x)}\,\Omega(-t')}{|t|^{n+\alpha}}\,dt \right. \\
&\quad \left. + \sum_{\nu=1}^{l} (-1)^\nu C_l^\nu \int_{R^n} k_\theta^\alpha(t)\,dt \int_{|x|>\varepsilon} \frac{\overline{\varphi(t-\nu x)}\,\Omega(-x')}{|x|^{n+\alpha}}\,dx \right\} \\
&= \lim_{\varepsilon \to 0} \int_{R^n} k_\theta^\alpha(x)(\mathbf{D}_{\Omega^{-1},\varepsilon}^\alpha \tilde{\varphi})(x)\,dx.
\end{aligned}$$

To get (6.5) it remains to justify passage to the limit under the integral sign. This can be done with the aid of the Lebesgue dominated convergence theorem with the estimate

$$|(\mathbf{D}_{\Omega^*,\varepsilon}^\alpha \varphi)(x)| \leq c(1+|x|)^{-n-\alpha} \tag{6.6}$$

taken into account (cf. Theorem 11), where c does not depend on ε.

Let us show that the functional G is continuous (i.e., that $G \in S'$). Suppose that $\varphi_m(x) \to 0$ as $m \to \infty$ in the topology of S. From (6.5) we have

$$|\langle G, \varphi_m \rangle| \leq \int_{|x|<R} |k_\theta^\alpha(x)(D_{\Omega^*}^\alpha \varphi_m)(x)|\,dx + \int_{|x|>R} |k_\theta^\alpha(x)(\mathbf{D}_{\Omega^*}^\alpha \varphi_m)(x)|\,dx.$$

The second term can be estimated on the basis of (6.6). In the first term $\mathbf{D}^{\alpha}_{\Omega\cdot}\varphi_m$ can be estimated directly after application of the formula

$$(\Delta'_t f)(x) \equiv \sum_{|k|=m} \sum_{\nu=0}^{l} \frac{(-\nu t)^l(-1)^\nu}{k!} C_l^\nu (D^k f)(x - \xi\nu t),$$

where $0 < \xi < 1$ and $l \geqslant m$ (written out for the noncentered difference). The corresponding computations are not complicated and are omitted.

2. Since $\langle \hat{G}, \varphi \rangle \overset{\text{def}}{=} (2\pi)^n \langle G, \tilde{\varphi} \rangle$, we have

$$\langle \hat{G}, \varphi \rangle = \int_{R^n} k^{\alpha}_{\theta}(x)(\mathbf{D}^{\alpha}_{\Omega\cdot}\hat{\tilde{\varphi}})(x)\,dx.$$

by virtue of (6.5). Since

$$\mathbf{D}^{\alpha}_{\Omega\cdot}\hat{\varphi} = (2\pi)^n F^{-1}[\mathfrak{D}^{\alpha}_{\Omega\cdot}(x)\varphi(-x)] = (2\pi)^n F^{-1}[\mathfrak{D}^{\alpha}_{\Omega}(-x)\varphi(-x)]$$

for $\varphi \in S(R^n)$, we have

$$\langle \hat{G}, \varphi \rangle = (2\pi)^n \int_{R^n} k^{\alpha}_{\theta}(x) F^{-1}\left[\mathfrak{D}^{\alpha}_{\Omega}(-x)\overline{\varphi(-x)}\right] dx.$$

Here we must carry over the Fourier transformation to the kernel $k^{\alpha}_{\theta}(x)$, i.e., prove the equality

$$\int_{R^n} \frac{\theta(x')}{|x|^{n-\alpha}} \hat{\psi}(x)\,dx = \int_{R^n} \mathcal{K}^{\alpha}_{\theta}(x)\psi(x)\,dx$$

for functions $\psi(x)$ of the form $\psi(x) = \mathfrak{D}^{\alpha}_{\Omega}(x)\overline{\varphi(x)}$, $\varphi(x) \in S(R^n)$. We established such an equality in Theorem 5 (with the footnote to that theorem taken into account). The condition $|\hat{\psi}(x)| \leqslant c|x|^{-b}$, $b > n - \alpha$, in that footnote holds by Theorem 11.

3. In (6.4) we choose a function $\varphi(x) \in C_0^\infty$ with support in the shell $\eta < |x| < N$. By carrying out direct estimates, it is not hard to get (with the smoothness of the kernel $k^{\alpha}_{\theta}(x')$ on Σ_{n-1} taken into account) that $|(\mathbf{D}^{\alpha}_{\Omega,\varepsilon}k^{\alpha}_{\theta})(x)| \leqslant c$ for $\eta < |x| < N$, where c depends on η and N but not on ε. Then it is possible to take the limit under the integral sign in (6.4), and, since $\langle G, \varphi \rangle = \varphi(0)$, we have

$$\int_{\eta<|x|<N} \varphi(x)(\mathfrak{D}^{\alpha}_{\Omega}k^{\alpha}_{\theta})(x)\,dx = 0.$$

From this we have (6.2), since $\varphi(x)$ is arbitrary.

2. Fundamental solutions of hypersingular operators. In essence, the preceding theorem gives the fundamental solution of a hypersingular operator.

REPHRASED THEOREM 19. *The kernel $k^{\alpha}_{\theta}(x)$ of the potential $K^{\alpha}_{\theta}\varphi$ is the fundamental solution of the hypersingular equation $\mathbf{D}^{\alpha}_{\Omega}f = g$, whose characteristic $\Omega(t')$ is associated with $\theta(x')$.*

The converse problem is interesting. Namely, suppose that we are given an HSI with nonsingular symbol. How can be find from its characteristic $\Omega(t')$ the characteristic $\theta(t')$ of some potential (2.1) for which $\Omega(t')$ is the associated characteristic?

That is, for the HS operator $\mathbf{D}^\alpha_\Omega$ is it possible to find the fundamental solution from its given characteristic? Knowing $\Omega(t')$, is it possible to compute $\theta(t')$ effectively? The next theorem answers these questions (for simplicity we consider C^∞-characteristics and use HSI's of neutral type, so that for $\alpha = 1, 3, 5, \ldots$ the characteristic $\Omega(t')$ is taken to be even).

THEOREM 19′. *Let $\Omega(\sigma) \in C^\infty(\Sigma_{n-1})$. If the symbol of the HSI $\mathbf{D}^\alpha_\Omega f$ is not singular on Σ_{n-1}, then there exists a characteristic $\theta(\sigma) \in C^\infty(\Sigma_{n-1})$ such that*

$$\mathbf{D}^\alpha_\Omega\left(\frac{\theta(x')}{|x|^{n-\alpha}}\right) = \delta(x), \qquad 0 < \alpha < n,$$

where $\theta(x')$ is computed for $\alpha \neq 1, 3, 5, \ldots$ by the formula

$$\theta(x') = \frac{\Gamma(n-\alpha)}{(2\pi)^n} f.p. \int_{\Sigma_{n-1}} \frac{d\sigma}{\mathfrak{D}^\alpha_\Omega(\sigma)(-ix'\cdot\sigma)^{n-\alpha}}. \tag{6.7}$$

A proof can be obtained by the expansion of $1/\mathfrak{D}^\alpha_\Omega(\sigma)$ in a series of spherical harmonics and use of the formula

$$\sum_{k,\mu} i^k \frac{\Gamma((k+\alpha)/2)}{\Gamma((k+n-\alpha)/2)} \varphi_{k\mu} Y_{k\mu}(x') = \frac{\Gamma(\alpha)}{2^\alpha \pi^{n/2}} \text{f.p.} \int_{\Sigma_{n-1}} \frac{\varphi(\sigma)\,d\sigma}{(-ix'\cdot\sigma)^\alpha}, \qquad \alpha \neq 1, 2, 3, \ldots,$$

where $\varphi(\sigma) = \sum_{k,\mu} \varphi_{k\mu} Y_{k\mu}(\sigma)$. This formula is true also for integers α if the summation on the left-hand side is over the k having the same parity as α.

We remark that, because of the inclusion of the differential operators in the scale of HS operators (obtained in Theorem 13), the formula (6.7) (as well as its analogue for $\alpha = 1, 3, 5, \ldots$ and the similar formulas for HSI's of even or odd types) contain the fundamental solutions of elliptic homogeneous differential operators with constant coefficients.

3. Integral representation of truncated hypersingular integrals.

THEOREM 20. *Let $f(x) = K^\alpha_\theta \varphi$ be a potential* (2.1) *with density $\varphi(t) \in L_p(R^n)$, $1 \leqslant p < n/\alpha$. Then*

$$(D^\alpha_{\Omega,\varepsilon} f)(x) = \int_{R^n} \mathcal{K}_{\Omega,\theta}(\xi)\varphi(x - \varepsilon\xi)\,d\xi, \tag{6.8}$$

where the kernel $\mathcal{K}_{\Omega,\theta}(\xi)$ has the form

$$\mathcal{K}_{\Omega,\theta}(\xi) = \frac{1}{d_{n,l}(\alpha)|\xi|^n} \int_{|t|>1/|\xi|} \frac{(\Delta^l_t k^\alpha_\theta)(\xi/|\xi|)}{|t|^{n+\alpha}} \Omega(t/|t|)\,dt \tag{6.9}$$

$$= \frac{1}{d_{n,l}(\alpha)|\xi|^n} \int_{|t|<|\xi|} \Omega(t/|t|) k^{l,\alpha}_\theta(\xi/|\xi|, t)\,dt, \tag{6.10}$$

where

$$k^{l,\alpha}_\theta(\xi', t) = \sum_{\nu=0}^{l} (-1)^\nu C^\nu_l |t - \nu\xi'|^{\alpha-n} \theta\left(\frac{|t|\xi' - \nu t'}{||t|\xi' - \nu t'|}\right) \tag{6.11}$$

or

$$k_\theta^{l,\alpha}(\xi',t)=\sum_{\nu=0}^{l}(-1)^\nu C_l^\nu\left|t-\left(\frac{l}{2}-\nu\right)\xi'\right|^{\alpha-n}\theta\left(\frac{|t|\xi'-(l/2-\nu)t'}{||t|\xi'-(l/2-\nu)t'|}\right) \tag{6.12}$$

corresponds to the cases of the noncentered or centered difference in the HSI $\mathbf{D}_\Omega^\alpha$.

PROOF. We have

$$(\mathbf{D}_{\Omega,\varepsilon}^\alpha f)(x)=\frac{1}{d_{n,l}(\alpha)}\int_{R^n}\varphi(\xi)\,d\xi\int_{|t|>\varepsilon}\frac{(\Delta_t^l k_\theta^\alpha)(x-\xi)}{|t|^{n+\alpha}}\Omega(t')\,dt. \tag{6.13}$$

Suppose for definiteness that the HSI is of neutral type. Proceeding with (6.13), we have

$$(\mathbf{D}_{\Omega,\varepsilon}^\alpha f)(x)=\frac{1}{d_{n,l}(\alpha)}\int_{R^n}\varphi(x-\xi)\,d\xi$$
$$\times\int_{|t|>\varepsilon}\frac{\Omega(t')}{|t|^{n+\alpha}}\sum_{\nu=0}^{l}(-1)^\nu C_e^\nu|\xi-\nu t|^{\alpha-n}\theta\left(\frac{\xi-\nu t}{|\xi-\nu t|}\right)dt.$$

From this, after the substitutions $t=|\xi|y$ and $\xi=\varepsilon\eta$, we get

$$(\mathbf{D}_{\Omega,\varepsilon}^\alpha f)(x)=\frac{1}{d_{n,l}(\alpha)}\int_{R^n}\frac{\varphi(x-\varepsilon\eta)}{|\eta|^n}d\eta$$
$$\times\int_{|y|>1/|\eta|}\frac{\Omega(y')}{|y|^{n+\alpha}}\sum_{\nu=0}^{l}(-1)^\nu C_l^\nu|\eta'-\nu y|^{\alpha-n}\theta\left(\frac{\eta'-\nu y}{|\eta'-\nu y|}\right)dy$$
$$=\frac{1}{d_{n,l}(\alpha)}\int_{R^n}\varphi(x-\varepsilon\eta)\left(\frac{1}{|\eta|^n}\int_{|y|>1/|\eta|}\frac{(\Delta_y^l k_\theta^\alpha)(\eta')}{|y|^{n+\alpha}}\Omega(y')\,dy\right)d\eta,$$

which coincides with (6.8) and (6.9). The representation (6.10) is obtained by the inversion $y=t/|t|^2$ with use of the equality $|\xi'|t|-\nu t'|=|t-\nu\xi'|$.

REMARK 19. We emphasize that the representation (6.8) was obtained for arbitrary summable characteristics $\theta(t')$ and $\Omega(t')$ on Σ_{n-1} that are not connected with each other in any way. We shall see below in subsection 5 that if $\Omega(t')$ is the characteristic associated with $\theta(t')$, then $\mathcal{K}_{\Omega,\theta}(\xi)$ is an averaging kernel.

4. The Fourier transform of the kernel of the representation.

LEMMA 11. *The Fourier transform of the kernel* $\mathcal{K}_{\Omega,\theta}(\xi)$ *can be computed by the formula*

$$\hat{\mathcal{K}}_{\Omega,\theta}(x)=\frac{\mathcal{K}_\theta^\alpha(x')}{d_{n,l}(\alpha)}\int_{|t|>|x|}\frac{(1-e^{ix'\cdot t})^l}{|t|^{n+\alpha}}e^{-i(\lambda/2)x'\cdot t}\Omega(t')\,dt, \tag{6.14}$$

where $\lambda=0$ *in the case of an HSI of neutral type, and* $\lambda=1$ *in the case of even or odd type. Also,*

$$\hat{\mathcal{K}}_{\Omega,\theta}(x)=\mathcal{K}_\theta^\alpha(x)\int_{\Sigma_{n-1}}\Omega(\sigma)A(x\cdot\sigma)\,d\sigma, \tag{6.15}$$

where

$$A(y) = \frac{1}{d_{n,l}(\alpha)} \sum_{k=0}^{l} (-1)^k C_l^k (-i\tilde{k}y)^\alpha \Gamma(-\alpha, -i\tilde{k}y), \qquad -1 \leqslant y \leqslant 1, \quad (6.16)$$

$\Gamma(-\alpha, z)$ *is the incomplete gamma function, and* $\tilde{k} = k$ *for* $\lambda = 0$, $\tilde{k} = l/2 - k$ *for* $\lambda = 1$.(6)

PROOF. Similarly to (4.11), we have, for the truncated HSI in terms of Fourier transforms, that

$$(\mathbf{D}^\alpha_{\Omega,\varepsilon} f)\hat{}(x) = \mathfrak{D}^\alpha_{\Omega,\varepsilon}(x) f(x),$$

where

$$\mathfrak{D}^\alpha_{\Omega,\varepsilon}(x) = \frac{1}{d_{n,l}(\alpha)} \int_{|t|>\varepsilon} \frac{(1 - e^{ix\cdot t})^l}{|t|^{n+\alpha}} \Omega(t') e^{-i(\lambda/2)x'\cdot t}\, dt.$$

Then, by (2.89),

$$(\mathbf{D}^\alpha_{\Omega,\varepsilon} f)\hat{}(x) = \mathcal{K}^\alpha_\theta(x) \mathfrak{D}^\alpha_{\Omega,\varepsilon}(x) \hat{\varphi}(x). \quad (6.17)$$

On the other hand, the representation (6.8) yields

$$(\mathbf{D}^\alpha_{\Omega,\varepsilon} f)\hat{}(x) = \hat{\mathcal{K}}_{\Omega,\theta}(x) \hat{\varphi}(x). \quad (6.18)$$

Comparing (6.17) and (6.18), we get that

$$\hat{\mathcal{K}}_{\Omega,\theta}(x) = \mathcal{K}^\alpha_\theta(x) \mathfrak{D}^\alpha_{\Omega,\varepsilon}(x)\big|_{\varepsilon=1},$$

which coincides with (6.14). The representation (6.15) follows from (6.14) by passage to polar coordinates with account taken of the formula 3.381.3 in [6].

5. The kernel of the representation as an averaging kernel in the case of associated characteristics.

THEOREM 21. *If* $\theta(x')$ *and* $\Omega(x')$ *are associated characteristics, then* $\mathcal{K}_{\Omega,\theta}(x)$ *is an averaging kernel*:

$$\mathcal{K}_{\Omega,\theta}(x) \in L_1(R^n) \quad \text{and} \quad \int_{R^n} \mathcal{K}_{\Omega,\theta}(x)\, dx = 1. \quad (6.19)$$

PROOF. By the smoothness of the functions $\Omega(t')$ and $\theta(t')$, we conclude from (6.10) that $\mathcal{K}_{\Omega,\theta}(x)$ is at least continuous in $R^n \setminus \{0\}$. The estimation for $|\xi| \to 0$ is simple and follows directly from (6.10):

$$|\mathcal{K}_{\Omega,\theta}(\xi)| \leqslant c/|\xi|^{n-\alpha}, \qquad |\xi| \to 0. \quad (6.20)$$

For $|\xi| \to 0$ we have

$$|\mathcal{K}_{\Omega,\theta}(\xi)| \leqslant c/|\xi|^{n+l^*-\alpha}, \qquad |\xi| \to \infty, \quad (6.21)$$

where $l^* = l$ for $l > \alpha$ and $l^* = l + 1$ for $2[\alpha/2] < l \leqslant \alpha$. (Recall that the case $2[\alpha/2] < l \leqslant \alpha$ is allowed for noncentered differences and even characteristics.) The estimate (6.21) is more difficult, and to obtain it we must now use the connection

(6) In the case of a centered difference of even order the term with index $k = l/2$ in the sum (6.16), which has a removable singularity, should be replaced by $\frac{1}{\alpha}(-1)^{l/2} C_l^{l/2} |x|^{-\alpha}$.

between the characteristics $\theta(t')$ and $\Omega(t')$: it is based on the annihilation property (6.2). Let us prove (6.21). By (6.2),

$$\mathcal{K}_{\Omega,\theta}(\xi) = -\frac{1}{d_{n,l}(\alpha)|\xi|^n}\int_{|t|<1/|\xi|}\frac{(\Delta_t^l k_\theta^\alpha)(\xi')}{|t|^{n+\alpha}}\Omega(t')\,dt \tag{6.22}$$

$$= -\frac{1}{d_{n,l}(\alpha)|\xi|^n}\int_{|t|>|\xi|}\Omega(t')k_\theta^{l,\alpha}(\xi',t)\,dt. \tag{6.23}$$

We introduce a function

$$W(s) = |t+s\xi'|^{\alpha-n}\theta\left(\frac{|t|\xi'-st'}{||t|\xi'-st'|}\right), \qquad 0\leqslant s\leqslant 1, \tag{6.24}$$

such that $k_\theta^{l,\alpha}(\xi',t)$ is a finite difference of the function $W(s)$ of a single variable with step $h=1$ at the point $s=0$:

$$k_\theta^{l,\alpha}(\xi',t) = (\Delta_1^l W)(0)$$

(for fixed values of ξ' and t). By the familiar identity

$$(\Delta_h^l W)(s) = \int_0^h\cdots\int_0^h W^{(l)}(s+s_1+\cdots+s_l)\,ds_1\cdots ds_l$$

we get that

$$k_\theta^{l,\alpha}(\xi',t) = W^{(l)}(\eta), \qquad 0<\eta<l. \tag{6.25}$$

Direct differentiation of $W(s)$ by the Leibniz rule with the homogeneity of $\theta(\sigma)$ taken into account gives

$$|k_\theta^{l,\alpha}(\xi',t)|\leqslant c/(1+|t|)^{n+l-\alpha}, \qquad |t|\to\infty, \tag{6.26}$$

where c does not depend on ξ'. Then (6.21) for $l>\alpha$ follows from (6.23) and the boundedness of $\Omega(t')$. But if $2[\alpha/2]<l\leqslant\alpha$, then these arguments do not suffice.

We make use of formulas for passing from a difference of order l to a difference of order $l+1$ (see [25], (1.4) and (1.5)). In [25] the kernel $\mathcal{K}_{\Omega,\theta}(\xi)$ was estimated at infinity for the case $\Omega\equiv\theta\equiv 1$. The estimate in [25] could be made because it was observed that the integration in (6.22) and (6.23) was actually carried out not over a ball but over a shell. This device does not work here, and we shall transform the kernel $\mathcal{K}_{\Omega,\theta}(\xi)$ with the aid of the indicated passage. Namely, we apply the identity

$$(\Delta_t^l f)(x) = (P_t^l f)(x) - \sum_{k=1}^{(l-1)/2}(\Delta_t^{l+1}f)(x-kt) - \frac{1}{2}(\Delta_t^{l+1}f)\left(x+\frac{l+1}{2}t\right), \tag{6.27}$$

which we obtained in [25], §1. Here $l=1,3,5,\ldots$, and the function $(P_t^l f)(x)$ is odd with respect to t. Using the evenness of $\Omega(t')$, which follows (see Remark 17) from the evenness of $\theta(t')$, which, in turn, is mandatory (see Remark 15) for an HSI of neutral type, we get

$$\mathcal{K}_{\Omega,\theta}(\xi) = \frac{1}{d_{n,l}(\alpha)}\sum_{\mu=1}^{(l+1)/2}\frac{c_\mu}{|\xi|^n}\int_{|t|<1/|\xi|}\frac{(\Delta_t^{l+1}k_\theta^\alpha)(\xi'-\mu t)}{|t|^{n+\alpha}}\Omega(t')\,dt, \tag{6.28}$$

where $c_\mu = 1$ for $\mu = 1, 2, \ldots, (l-1)/2$, and $c_{(l+1)/2} = \frac{1}{2}$. Let $I_\mu(\xi)$ be the generic term of the sum in (6.28). After the substitution $t = \tau/|\tau|^2$ we obtain

$$I_\mu(\xi) = \frac{1}{|\xi|^n} \int_{|\tau|>|\xi|} \Omega(\tau') \overset{*}{k}{}_\theta^{l+1,\alpha}(\xi', \tau)\, d\tau, \tag{6.29}$$

where

$$\overset{*}{k}{}_\theta^{l+1,\alpha}(\xi', \tau) = \sum_{\nu=0}^{l+1} (-1)^\nu C_{l+1}^\nu |\tau - (\mu+\nu)\xi'|^{\alpha-n} \theta\left(\frac{\xi' - (\mu+\nu)(\tau/|\tau|^2)}{|\xi' - (\mu+\nu)(\tau/|\tau|^2)|} \right).$$

Then the estimates are performed similarly to (6.24)–(6.26) by introduction of the function

$$W(s) = |t - (\mu+s)\xi'|^{\alpha-n} \theta\left(\frac{\xi' - (\mu+s)(\tau/|\tau|^2)}{|\xi' - (\mu+s)(\tau/|\tau|^2)|} \right), \qquad 0 \leqslant s \leqslant 1;$$

here instead of (6.26) we get that

$$|\overset{*}{k}{}_\theta^{l,\alpha}(\xi', t)| \leqslant c/(1+|t|)^{n+l+1-\alpha},$$

and then (6.21) for $2[\alpha/2] < l \leqslant \alpha$ follows from (6.28) and (6.29).

The estimates (6.20) and (6.21) imply that $\mathcal{K}_{\Omega,\theta}(\xi) \in L_1(R^n)$. It remains to show that the kernel is normalized. For this, take $\varphi(x) \in \Phi(R^n)$ in (6.8) and let ε go to zero in (6.8). We get

$$(\mathbf{D}_\Omega^\alpha f)(x) = \varphi(x) \int_{R^n} \mathcal{K}_{\Omega,\theta}(\xi)\, d\xi.$$

It remains to use the fact that $(\mathbf{D}_\Omega^\alpha f)(x) = \varphi(x)$ for $\varphi(x) \in \Phi(R^n)$ (see Theorems 15, 17, and 17′). Theorem 21 is proved.

6. Convergence in L_p of the truncated hypersingular integrals and inversion of potentials in the framework of L_p-spaces. On the basis of Theorem 20 we now arrive at the fact that the truncated HSI $\mathbf{D}_{\Omega,\varepsilon}^\alpha f$ converges (in the L_p-norm) as $\varepsilon \to 0$ on functions representable by potentials with p-summable densities, and it generates an operator inverse to the potential $K_\theta^\alpha \varphi$ in the framework of the L_p-spaces. Namely,

THEOREM 22. *Suppose that $f(x) = K_\theta^\alpha \varphi$, $1 \leqslant p < n/\alpha$, and that conditions* (3.1) *and* (5.2) *hold for the characteristic $\theta(x')$. Then*

$$\lim_{\varepsilon \to 0} \|\mathbf{D}_{\Omega,\varepsilon}^\alpha f - \varphi\|_p = 0, \tag{6.30}$$

where $\Omega(x')$ is the characteristic associated with $\theta(x')$.

PROOF. By (6.8) and (6.19),

$$(\mathbf{D}_{\Omega,\varepsilon}^\alpha f)(x) - \varphi(x) = \int_{R^n} \mathcal{K}_{\Omega,\theta}(\xi)[\varphi(x - \varepsilon\xi) - \varphi(x)]\, d\xi$$

and (6.30) is obtained from this by using the Minkowski inequality with a subsequent passage to the limit under the integral sign justified by the Lebesgue dominated convergence theorem. The theorem is proved.

Thus, the hypersingular integral $\mathbf{D}^\alpha_\Omega f$ with the associated characteristic inverts the potential K^α_θ in the framework of the L_p-spaces,

$$\mathbf{D}^\alpha_\Omega K^\alpha_\theta \varphi \equiv \varphi, \qquad \varphi \in L_p(R^n), \quad 1 \leqslant p < n/\alpha,$$

if convergence of the hypersingular integral is understood in the sense of convergence in $L_p(R^n)$:

$$\mathbf{D}^\alpha_\Omega f \overset{\text{def}}{=} \lim_{\substack{\varepsilon \to 0 \\ (L_p)}} \mathbf{D}^\alpha_{\Omega,\varepsilon} f.$$

BIBLIOGRAPHY

1. A. Erdélyi et al., *Higher transcendental functions*. Vol. 2, McGraw-Hill, 1953.

2. F. R. Gantmakher, *The theory of matrices*, 2nd ed., "Nauka", Moscow, 1966; English transl. of 1st ed., Vols. 1, 2, Chelsea, New York, 1959.

3. F. D. Gakhov, *Boundary-value problems*, 3rd ed., "Nauka", Moscow, 1977; English transl. of 2nd ed., Pergamon Press, Oxford, and Addison-Wesley, Reading, Mass., 1966.

4. I. M. Gel'fand and G. E. Shilov, *Generalized functions*. Vol. 1: *Operations on them*, 2nd ed., Fizmatgiz, Moscow, 1959; English transl. of 1st ed., Academic Press, 1964.

5. ______, *Generalized functions*. Vol. 2: *Spaces of test functions*, Fizmatgiz, Moscow, 1958; English transl., Academic Press and Gordon and Breach, New York, 1968.

6. I. S. Gradshteĭn and I. M. Ryzhik, *Tables of integrals, series and products*, 4th ed., Fizmatgiz, Moscow, 1963; English transl., Academic Press, 1965.

7. A. Zygmund, *Trigonometric series*, 2nd rev. ed., Vol. I, Cambridge Univ. Press, 1959.

8. P. I. Lizorkin, *Generalized Liouville differentiation and the function spaces $L^r_p(E^n)$. Imbedding theorems*, Mat. Sb. **60** (**102**) (1963), 325–353. (Russian)

9. ______, *Generalized Liouville differentiation and the method of multipliers in the theory of imbeddings of classes of differentiable functions*, Trudy Mat. Inst. Steklov. **105** (1969), 89–167; English transl. in Proc. Steklov Inst. Math. **105** (1969).

10. ______, *Description of the spaces $L^r_p(R^n)$ in terms of singular difference integrals*, Mat. Sb. **81** (**123**) (1970), 79–91; English transl. in Math. USSR Sb. **10** (1970).

11. S. G. Mikhlin, *Multidimensional singular integrals and integral equations*, Fizmatgiz, Moscow, 1962; English transl., Pergamon Press, 1965.

12. È. L. Radzhabov, *Certain hypersingular integral operators*, Izv. Akad. Nauk Tadzhik. SSR. Otdel. Fiz.-Mat. i Geolog.-Khim. Nauk **1974**, no. 2 (52), 17–25. (Russian)

13. ______, *Composition of double hypersingular integrals*, Dokl. Akad. Nauk Tadzhik. SSR **17** (1974), no. 11, 8–11. (Russian)

14. ______, *Pseudodifferential operators with increasing symbols in the space $P_{s,m}(R^n)$*, Izv. Akad. Nauk Tadzhik. SSR. Otdel. Fiz.-Mat. i Geolog.-Khim. Nauk **1975**, no. 1 (55), 3–11. (Russian)

15. ______, *A class of pseudodifferential operators that are representable in the form of a hypersingular integral*, Izv. Akad. Nauk Tadzhik. SSR. Otdel. Fiz.-Mat. i Geolog.-Khim. Nauk **1977**, no. 3 (65), 3–12. (Russian)

16. B. S. Rubin, *On operators of potential type in weighted spaces on an arbitrary contour*, Dokl. Akad. Nauk SSSR **207** (1972), 300–303; English transl. in Soviet Math. Dokl. **13** (1972).

17. ______, *Operators of potential type on a segment of the real line*, Izv. Vyssh. Uchebn. Zaved. Matematika **1973**, no. 6 (133), 73–81. (Russian)

18. ______, *Fractional integrals in Hölder spaces with weight, and operators of potential type*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **9** (1974), 308–324. (Russian)

19. ______, *The Noetherianness of operators of potential type in weighted spaces of functions p-summable with weight*, Izv. Vyssh. Uchebn. Zaved. Matematika **1975**, no. 8 (159), 81–90; English transl. in Soviet Math. (Iz. VUZ) **19** (1975).

20. S. G. Samko, *A generalized Abel equation and fractional integration operators*, Differentsial'nye Uravneniya **4** (1968), 298–314; English transl. in Differential Equations **4** (1968).

21. ______, *Noether's theory for the generalized Abel integral equation*, Differentsial'nye Uravneniya **4** (1968), 315–326; English transl. in Differential Equations **4** (1968).

22. ______, *Abel's generalized integral equation on the line*, Izv. Vyssh. Uchebn. Zaved. Matematika **1970**, no. 8 (99), 83–93. (Russian)

23. ______, *The space $I^\alpha(L_p)$ of fractional integrals, and operators of potential type*, Izv. Akad. Nauk Armyan. SSR Ser. Mat. **8** (1973), 359–383. (Russian)

24. ______, *The spaces $L^\alpha_{p,r}(R^n)$, and hypersingular integrals*, Vestsī Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk **1976**, no. 2, 34–41. (Russian)

25. ______, *On spaces of Riesz potentials*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 1143–1172; English transl. in Math. USSR Izv. **10** (1976).

26. ______, *Generalized Riesz potentials*, Gamoqeneb. Math. Inst. Sem. Mohsen. Anotacie Vyp. **11** (1976), 35–44. (Russian)

27. ______, *Generalized Riesz potentials and hypersingular integrals, their symbols and inversion*, Dokl. Akad. Nauk SSSR **232** (1977), 528–531; English transl. in Soviet Math. Dokl. **18** (1977).

28. ______, *On the description of the image $I^\alpha(L_p)$ of Riesz potentials*, Izv. Akad. Nauk Armyan. SSR Ser. Math. **12** (1977), 329–334. (Russian)

29. ______, *The spaces $L^\alpha_{p,r}(\mathbf{R}^n)$ and hypersingular integrals*, Studia Math. **61** (1977), 193–230. (Russian)

30. ______, *A criterion for the harmonicity of polynomials*, Differentsial'nye Uravneniya **14** (1978), 938–939; English transl. in Differential Equations **14** (1978).

31. ______, *The Fourier transform of the functions $Y_m(x/|x|)|x|^{-n+\alpha}$*, Izv. Vyssh. Uchebn. Zaved. Matematika **1978**, no. 7 (194), 73–78; English transl. in Soviet Math. (Iz. VUZ) **22** (1978).

32. ______, *Hypersingular integrals with homogeneous characteristics*, Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy **5/6** (1978), 235–249. (Russian)

33. Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.

34. Elias M. Stein and Guido Weiss, *Introduction of Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N. J., 1971.

35. J. F. Treves [François Trèves], *Lectures on linear partial differential equations with constant coefficients*, Notas de Math., No. 27, Inst. Mat. Pura Apl. Conselho Nac. Pesquisas, Rio de Janeiro, 1961.

36. R. E. Edwards, *Functional analysis. Theory and applications*, Holt, Reinhart and Winston, New York, 1965.

37. S. Bochner, *Theta relations with spherical harmonics*, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 804–808.

38. Michael J. Fisher, *Some generalizations of the hypersingular integral operators*, Studia Math. **47** (1973), 95–121.

39. Carl S. Herz, *On the mean inversion of Fourier and Hankel transforms*, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 996–999.

40. Claus Müller, *Spherical harmonics*, Lecture Notes in Math., Vol. 17, Springer-Verlag, 1966.

41. Umberto Neri, *Singular integrals*, Lecture Notes in Math., Vol. 200, Springer-Verlag, 1971.

42. Victor L. Shapiro, *Topics in Fourier and geometric analysis*, Mem. Amer. Math. Soc. No. 39 (1961).

43. L. Schwartz, *Théorie des distributions*. Tome II, Actualités Sci. Indust., No. 1122, Hermann, Paris, 1951.

44. Walter Trebels, *Generalized Lipschitz conditions and Riesz derivatives on the space of Bessel potentials L^p_α*. I: *Conditions for elements of L^p_α and their Riesz transforms $0 < \alpha \leq 2$*, Applicable Anal. **1** (1971), 75–99.

45. ______, *Imbedding theorems for spaces of hypersingular integrals and Bessel potentials*, J. Approximation Theory **6** (1972), 202–214.

46. Richard L. Wheeden, *On hypersingular integrals and Lebesgue spaces of differentiable functions*. I, Trans. Amer. Math. Soc. **134** (1968), 421–435.

47. ______, *On hypersingular integrals and Lebesgue spaces of differentiable functions*. II, Trans. Amer. Math. Soc. **139** (1969), 37–53.

48. ______, *On hypersingular integrals and certain spaces of locally differentiable functions*, Trans. Amer. Math. Soc. **146** (1969), 211–230.

49. ______, *A note on a generalized hypersingular integral*, Studia Math. **44** (1972), 17–26.

Translated by H. H. McFADEN