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ON SPACES OF RIESZ POTENTIALS

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Abstract. In connection with problems which arise in the theory of integral equations of the first kind with a potential-type kernel we investigate the space of Riesz potentials $I^{\alpha}(L_p) = \{f = K^{\alpha}\varphi;\ \varphi \in L_p(R^n),\ 1 < p < n/\alpha\}$, where K^{α} is the Riesz integration operator ($\widehat{K^{\alpha}\varphi}(x) = |x|^{-\alpha}\hat{\varphi}(x)$). We give a description of the space $I^{\alpha}(L_p)$ in terms of differences of singular integrals, establish a theorem on denseness of $C_0^{\infty}(R^n)$ in $I^{\alpha}(L_p)$, and indicate a "weight" invariant description of $I^{\alpha}(L_p)$.

Bibliography: 44 titles.

The theory of Sobolev spaces of Bessel potentials, which are related to multiplication by $(1 + |x|^2)^{\alpha/2}$ in Fourier transforms and which turn out to be very convenient in applications, is well known ([1], [7]–[9], [14], [27], [28], [44], etc.). Some problems, however, reduce (see, for example, [18]–[20]) to the spaces $I^{\alpha}(L_p)$ of Riesz potentials, which are related to multiplication by $|x|^{\alpha}$ in Fourier transforms and which consist of functions $f(x)$ whose derivatives $D^{\beta}f$ are summable in R^n to power $p_{|\beta|}$ which is independent of $|\beta|$. Our goal is a study of spaces of Riesz potentials in R^n. In this paper we consider the following basic questions: 1) a description of the spaces $I^{\alpha}(L_p)$ in terms of the Riesz derivatives $\mathrm{D}_l^{\alpha}f$, which are the singular difference integrals introduced by E. M. Stein [42] for $l = 1$, $0 < \alpha < 2$, and in their general form by P. I. Lizorkin [8]–[10], and which are the multidimensional analog of the fractional derivatives of Marchaud [38] (in the case of spaces of Bessel potentials, such a description is given in [9]); 2) the denseness of C_0^{∞} in $I^{\alpha}(L_p)$; 3) bounds of the type $\|D^{\beta}f\|_q \leqslant c\|\mathrm{D}_l^{\alpha}f\|_p$, $q \leqslant p$, $0 < |\beta| \leqslant \alpha$, $f \in L_{p,r}^{\alpha}$.

We note that for consideration of Riesz derivatives $\mathrm{D}_l^{\alpha}f$ we may successfully weaken the usual condition $l > \alpha$ to $l > 2[\alpha/2]$; moreover, it turns out that the case where α is an odd positive integer is to some extent exceptional, requiring us to take only $l = \alpha$.

In §1 there are auxiliary assertions (we call attention to one of them, on "annihilation" of the Riesz derivatives of odd integral order for $l > \alpha$). In §2, for spaces of Riesz potentials we construct an integral representation, using which in §3 we give a description of the spaces $I^{\alpha}(L_p)$ in terms of Riesz differentiation. In §3 we also establish certain imbeddings of spaces. In §4 we clean up the question of existence of weak derivatives of orders $|\beta| \leqslant \alpha$ and their representation by Riesz derivatives. §5 contains a theorem on the denseness of

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C_0^∞ in $I^\alpha(L_p)$. In view of the above-indicated "annihilation" of the derivatives $D_l^{2m+1}f$ for $l > 2m+1$, it is necessary here to study the case $\alpha = 2m+1$ separately. And, finally, §6 contains yet another variant of the description of the spaces $I^\alpha(L_p)$, in which information on a function $f(x)$ is given in terms of weights.

We should point out the work of C. S. Herz [34], in which, in particular, the operation of integration was investigated in the scope of the spaces $\Lambda^\alpha_{a,p}$, which are related to Besov-Taibleson spaces.

We remark that for consideration of weak derivatives in §4 we use in an essential way the space Φ of functions $f(x) \in S$ which are orthogonal to the polynomials and invariant relative to Riesz integration; this space was introduced by P. I. Lizorkin in [7] and [8].

And, finally, we note that a description of the space $I^\alpha(L_p)$ in the one-dimensional case was given in [18] and [20].

NOTATION. R^n denotes euclidean space, $x = (x_1, \dots, x_n) \in R^n$, $\vec{j} = (1, 0, \dots, 0)$;

$$D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}},$$

$\beta = (\beta_1, \dots, \beta_n)$ is a multi-index, $|\beta| = \beta_1 + \dots + \beta_n$;

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_n^2};$$

$$dx = dx_1 \dots dx_n, \quad (f, \varphi) = \int_{R^n} f(x)\varphi(x)\,dx; \quad (\tau_t f)(x) = f(x-t), \quad t \in R^n;$$

$$(\Delta_t^l f)(x) = \sum_{k=0}^{l} (-1)^k C_l^k f(x - kt) = (I - \tau_t)^l f;$$

$$\omega_p^l(f, h) = \|(\Delta_h^l f)(\cdot)\|_p, \quad \|f\|_p = \|f\|_{L_p(R^n)}, \quad p' = \frac{p}{p-1};$$

$$F\varphi = \hat{\varphi}(x) = \int_{R^n} e^{i(x,t)}\varphi(t)\,dt, \quad p_\alpha = \frac{np}{n - \alpha p} \quad (\alpha p < n);$$

$$L_p(\alpha; R^n) = \left\{ f: \int_{R^n} \frac{|f(x)|^p}{(1+|x|)^{\alpha p}}\,dx < \infty \right\},$$

$x = \mathrm{rot}_h y$ is the rotation in R^n which sends $y \in R^n$ to $x \in R^n$ so that $h/|h| = \mathrm{rot}_h \vec{j}$; $C_0^\infty = C_0^\infty(R^n)$ is the class of infinitely differentiable functions with compact support in R^n, and S is the class of infinitely differentiable functions which decrease at infinity faster than any power (see [2]).* Φ is the subspace of S consisting of functions which are orthogonal to the polynomials [7], [8]; a (continuous) imbedding $Z_1 \subset Z_2$, $\|\cdot\|_{Z_2} \leq c\|\cdot\|_{Z_1}$, of two normed spaces will be denoted, as usual, by $Z_1 \longrightarrow Z_2$; and, finally,

* *Translator's note.* The reference here is apparently to §1.10 in Chapter 1 of [2], where S is used to denote the class of infinitely differentiable functions in R^n which together with their derivatives approach zero more rapidly than any power of $1/|x|$ as $|x| \to \infty$.

$$\mathcal{A}_l(\alpha) = \sum_{k=1}^{l} (-1)^{k-1} C_l^k k^\alpha, \tag{1}$$

$$c_n = \pi^{\frac{n+1}{2}} \Gamma^{-1}\left(\frac{n+1}{2}\right), \quad \gamma_n(\alpha) = \pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right) \Gamma^{-1}\left(\frac{n-\alpha}{2}\right), \tag{2}$$

$$d_{n,l}(\alpha) = \begin{cases} \dfrac{\pi^{\frac{n}{2}+1}}{2^\alpha \Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{n+\alpha}{2}\right)} \dfrac{\mathcal{A}_l(\alpha)}{\sin\frac{\alpha\pi}{2}}, & \alpha \neq 2, 4, 6, \ldots, \\ \dfrac{(-1)^{\frac{\alpha}{2}} \pi^{\frac{n}{2}} 2^{1-\alpha}}{\Gamma\left(1+\frac{\alpha}{2}\right) \Gamma\left(\frac{n+\alpha}{2}\right)} \dfrac{d}{d\alpha} \mathcal{A}_l(\alpha), & \alpha = 2, 4, 6, \ldots. \end{cases} \tag{3}$$

§1. Auxiliary assertions

The Riesz potential

$$(K^\alpha \varphi)(x) = \frac{1}{\gamma_n(\alpha)} \int_{R^n} \frac{\varphi(t)\, dt}{|x-t|^{n-\alpha}}, \quad x \in R^n, \quad 0 < \alpha < n, \tag{1.1}$$

is defined, according to a theorem of S. L. Sobolev [22], on all of $L_p(R^n)$ for $1 < p < n/\alpha$. It is well known ([6], [23], etc.) that

$$\widehat{K^\alpha \varphi}(x) = |x|^{-\alpha} \hat{\varphi}(x) \tag{1.2}$$

for the choice of $\gamma_n(\alpha)$ indicated in (2) (at least for $\varphi \in S$). Following Lizorkin [9], we put

$$(T_l^\alpha f)(x) = \int_{R^n} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}} dt, \quad \alpha > 0. \tag{1.3}$$

(In [9], for the definition of $T_l^\alpha f$ a fourth order centered difference was used instead of $(\Delta_t^l f)(x)$.) It is obvious that the integral in (1.3) converges (absolutely) for $l > \alpha$ for sufficiently smooth functions. We show that it converges conditionally for $l > 2[\alpha/2]$. (In other words, for l we can take the odd integer next to α.) We can see this immediately for $f \in S$, and below, in §3, we prove it for all of $I^\alpha(L_p)$. We shall assume that l is odd, since if $l > 2[\alpha/2]$ is even, then $l > \alpha$. We have the identity ($l = 1, 3, 5, \ldots$)

$$(\tau-1)^l = \mathcal{P}_l(\tau) + \left(\frac{1}{\tau} + \frac{1}{\tau^2} + \ldots + \frac{1}{\tau^{\frac{l-1}{2}}} + \frac{\frac{1}{2}}{\tau^{\frac{l+1}{2}}}\right)(\tau-1)^{l+1}, \tag{1.4}$$

where

$$\mathcal{P}_l(\tau) = \frac{1}{2}(\tau-1)^l (\tau+1) \tau^{-\frac{l+1}{2}}$$

which is anti-invariant under reciprocals: $\mathcal{P}_l(1/\tau) = -\mathcal{P}_l(\tau)$. The following identity in finite dimensions corresponds to (1.4):

$$(\Delta_t^l f)(x) = (P_t^l f)(x) - \sum_{k=1}^{\frac{l-1}{2}} (\Delta_t^{l+1} f)(x+kt) - \frac{1}{2}(\Delta_t^{l+1} f)\left(x + \frac{l+1}{2}t\right),$$

where the function $(P_t^l f)(x)$ is odd in t. In such a case the limit

$$\lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} \frac{(\Delta_t^l f)(x)}{|t|^{n+\alpha}}\,dx = -\sum_{k=1}^{\frac{l-1}{2}} \int_{R^n} \frac{(\Delta_t^{l+1} f)(x+kt)}{|t|^{n+\alpha}}\,dt - \frac{1}{2}\int_{R^n} \frac{(\Delta_t^{l+1} f)\left(x+\frac{l+1}{2}t\right)}{|t|^{n+\alpha}}\,dt \quad (1.5)$$

exists, since $l + 1 > \alpha$.

It is not difficult to show that

$$\widehat{(T_l^\alpha f)}(x) = d_{n,l}(\alpha)\,|x|^\alpha \hat{f}(x), \quad f \in S, \quad (1.6)$$

where $d_{n,l}(\alpha)$ is the constant (3). It is calculated in the Appendix to this paper. Here we encounter a rather unexpected property: $d_{n,l}(\alpha) = 0$ for $\alpha = 1, 3, 5, \ldots$ and $l > \alpha$ at the expense of the coefficient $\mathfrak{A}_l(\alpha)$, which vanishes for $\alpha = 1, 2, \ldots, l-1$. The question arises naturally as to the vanishing of $d_{n,l}(\alpha)$ for other values of α. We have the following result.

LEMMA 1. *Let $\alpha > 0$. The roots of the equation $\mathfrak{A}_l(\alpha) = 0$ consist precisely of the numbers $\alpha = 1, 2, \ldots, l-1$.*

PROOF. We have the equality

$$\mathcal{A}_l(m) = -\left(x\frac{d}{dx}\right)^m (1-x)^l\big|_{x=1}, \quad m = 1, 2, \ldots, l, \quad (1.7)$$

whence it follows that $\mathfrak{A}_l(m) = 0$, $m = 1, \ldots, l-1$, but $\mathfrak{A}_l(l) = (-1)^{l+1} l! \neq 0$. (We note that the sum $\mathfrak{A}_l(m)$ for positive integers m is well known in combinatorial analysis; see [16], p. 50, or [37]; it was used in [44], p. 84.) It remains to show that $\mathfrak{A}_l(\alpha) \neq 0$ if $\alpha \neq 1, 2, \ldots, l-1$. For this we find the analog of (1.7) for nonintegral α. The natural means for this is the Hadamard fractional integro-differentiation $H^\alpha = (xd/dx)^\alpha$ (see [32] or [26], formula (23.17.1)):

$$H^\mu \varphi = \frac{1}{\Gamma(\mu)} \int_0^x \frac{\varphi(t)\,dt}{t\left(\ln\frac{x}{t}\right)^{1-\mu}}, \quad \mu > 0, \quad (1.8)$$

$$H^\mu \varphi = \left(x\frac{d}{dx}\right)^{[-\mu]+1} H^{1-\{-\mu\}}\varphi = H^{1-\{-\mu\}}\left(t\frac{d}{dt}\right)^{[-\mu]+1}\varphi, \quad \mu < 0 \quad (1.8')$$

($\varphi(0) = 0$, $x > 0$), which has the property

$$H^\mu(x^\lambda) = \frac{x^\lambda}{\lambda^\mu}, \quad \lambda > 0, \quad -\infty < \mu < \infty. \quad (1.9)$$

It follows immediately from (1.9) that

$$H^{\mu}\left[(1-t)^{l}-1\right](x)=\sum_{k=1}^{l}(-1)^{k}C_{l}^{k}k^{-\mu}x^{k},$$

from which we obtain

$$\mathcal{A}_{l}(\alpha)=-H^{-\alpha}\left[(1-t)^{l}-1\right](x)\big|_{x=1}, \tag{1.10}$$

and then

$$\mathcal{A}_{l}(\alpha)=-\frac{l}{\Gamma(1-\alpha)}\int_{0}^{1}\frac{(1-t)^{l-1}dt}{\left(\ln\frac{1}{t}\right)^{\alpha}}, \quad 0<\alpha<l, \quad \alpha\neq 1,2,\ldots,l-1. \tag{1.11}$$

In fact, for $0<\alpha<1$, (1.11) follows from (1.10) by (1.8′), and for the remaining values of α it can be obtained by analytic continuation with respect to α. It follows from (1.11) that $\mathcal{A}_{l}(\alpha)\neq 0$ for the values of α indicated in (1.11). In addition,

$$(-1)^{l-1}\mathcal{A}_{l}(\alpha)>0, \quad \alpha>l, \tag{1.12}$$

and, consequently, $\mathcal{A}_{l}(\alpha)\neq 0$ also for $\alpha>l$. It is not difficult to obtain a proof of (1.12) by induction, using the easily verified recursion formula

$$\frac{1}{l+1}\mathcal{A}_{l+1}(\alpha+1)=\mathcal{A}_{l+1}(\alpha)-\mathcal{A}_{l}(\alpha).$$

We shall not dwell on this.

With regard to (1.6) and Lemma 1 we define the Riesz derivative $\mathrm{D}^{\alpha}f$ by the equation

$$(\mathrm{D}_{l}^{\alpha}f)(x)=\frac{1}{d_{n,l}(\alpha)}\int_{R^{n}}\frac{(\Delta_{t}^{l}f)(x)}{|t|^{n+\alpha}}dt=\frac{1}{d_{n,l}(\alpha)}(T_{l}^{\alpha}f)(x), \tag{1.13}$$

where l is any positive integer greater than $2[\alpha/2]$ if $\alpha\neq 1,3,5,\ldots$, and $l=\alpha$ if $\alpha=1,3,5,\ldots$. For the choice of $d_{n,l}(\alpha)$ indicated in (3) the derivatives (1.13) are independent of l, since

$$\widehat{\mathrm{D}_{l}^{\alpha}f}(x)=|x|^{\alpha}f(x), \quad f\in S, \tag{1.14}$$

and we shall sometimes write $\mathrm{D}^{\alpha}f$ instead of $\mathrm{D}_{l}^{\alpha}f$, understanding in what follows the integral in (1.13) as conditionally converging in $L_{p}(R^{n})$:

$$(\mathrm{D}_{l}^{\alpha}f)(x)\overset{\text{def}}{=}\lim_{\substack{\varepsilon\to 0\\(L_{p})}}(\mathrm{D}_{l,\varepsilon}^{\alpha}f)(x), \quad l>2\left[\frac{\alpha}{2}\right], \tag{1.15}$$

where $\mathrm{D}_{l,\varepsilon}^{\alpha}f$ is the "truncated" Riesz derivative:

$$\mathrm{D}_{l,\varepsilon}^{\alpha}f=\frac{1}{d_{n,l}(\alpha)}T_{l,\varepsilon}^{\alpha}f=\frac{1}{d_{n,l}(\alpha)}\int_{|t|>\varepsilon}\frac{(\Delta_{t}^{l}f)(x)}{|t|^{n+\alpha}}dt. \tag{1.16}$$

In view of the sudden emergence of the "annihilation" of the Riesz derivatives of odd integral order:

$$T_l^\alpha f \equiv 0, \quad l > \alpha = 2m+1, \tag{1.17}$$

we shall indicate yet another proof of (1.17), without passing to the Fourier transform; the proof is limited to the case $\alpha = 1$. We have the following result.

LEMMA 2. *Let $0 < \alpha < 2$ and let $f(x)$ be such that $T_1^\alpha f$ exists as the limit* (1.15) *for $l = 1$. Then this is so also for $l = 2, 3, \ldots$, and*

$$T_l^\alpha f = \mathcal{A}_l(\alpha)\, T_1^\alpha f. \tag{1.18}$$

In fact, since

$$(\Delta_t^l f)(x) = \sum_{k=0}^{l-1} (-1)^k C_{l-1}^k [f(x-kt) - f(x-kt-t)],$$

we have

$$T_{l,\varepsilon}^\alpha f = T_{1,\varepsilon}^\alpha f + \sum_{k=1}^{l-1} (-1)^k C_{l-1}^k \left(\int_{|t|>\varepsilon} \frac{f(x-kt)-f(x)}{|t|^{n+\alpha}}\, dt + \int_{|t|>\varepsilon} \frac{f(x)-f(x-kt-t)}{|t|^{n+\alpha}}\, dt \right).$$

Making the substitution $kt \longrightarrow t$ in the first integral and $(k+1)t \longrightarrow t$ in the second, we obtain

$$T_{l,\varepsilon}^\alpha f = T_{1,\varepsilon}^\alpha f + \sum_{k=1}^{l-1} (-1)^k C_{l-1}^k [(k+1)^\alpha T_{1,\frac{\varepsilon}{k+1}}^\alpha f - k^\alpha T_{1,\frac{\varepsilon}{k}}^\alpha f],$$

and hence

$$T_l^\alpha f = \left[1 + \sum_{k=1}^{l-1} (-1)^k C_{l-1}^k ((k+1)^\alpha - k^\alpha) \right] T_1 f,$$

which coincides with (1.18) and proves Lemma 2 and, together with it, also (1.17) for $\alpha = 1$.

We note that $\mathrm{D}^\alpha f = (-\Delta)^m f$, $f \in S$, for $\alpha = 2m$, $m = 1, 2, \ldots$, by (1.14), and in general

$$\mathrm{D}^\alpha f = \mathrm{D}^{\alpha-2m}(-\Delta)^m f = (-\Delta)^m \mathrm{D}^{\alpha-2m} f, \quad f \in S, \tag{1.19}$$

for $\alpha \geqslant 2m$. In §4 we shall show the validity of (1.19) also in the setting of spaces of Riesz potentials.

In what follows we use the following lemmas, which can be easily proved.

LEMMA 3. *If $a(x) \in C_0^\infty$, then*: 1) *the representation*

$$a(x) - a(x-t) = \sum_{k=1}^{n} \frac{t_k}{1+|t|} \frac{\partial}{\partial x_k}\, a(x) + \psi(x, t) \tag{1.20}$$

holds, where $|\psi(x, t)| \leqslant c|t|^2(1+|x|)^{-2}(1+|x-t|)^{-2}$; 2)

$$|(\Delta_t^l a)(x)| \leqslant c \frac{|t|^l}{\prod_{k=0}^{l} (1+|x-kt|)}, \tag{1.21}$$

$$\omega_p^l(a,t) \leqslant c\left(\frac{|t|}{1+|t|}\right)^l, \quad p \geqslant 1, \quad l=1,2,3,\ldots. \tag{1.22}$$

LEMMA 4. *The integral*

$$\mathcal{J}_l^{\mu}(t) = \int_{R^n} \frac{dx}{\prod_{k=0}^{l} (1+|x-kt|)^{\mu}}, \quad l=1,2,3,\ldots, \tag{1.23}$$

admits the bound $\mathcal{J}_l^{\mu}(t) \leqslant c(1+|t|)^{n-(l+1)\mu+\epsilon}$, *where* $\epsilon > 0$ *is arbitrarily small and* $c = c(\epsilon)$, *for* $\mu > n/(l+1)$, *and the bound* $\mathcal{J}_l^{\mu}(t) \leqslant c(1+|t|)^{-\mu}$ *for* $\mu > n/l$.

In §4 we shall rely on some properties of the Riesz transform $\mathbf{R} = \{R_1, \ldots, R_n\}$,

$$(R_j f)(x) = \frac{1}{c_n} \lim_{\substack{\varepsilon \to 0 \\ (L_p)}} \int_{|t|>\varepsilon} \frac{t_j}{|t|^{n+1}} f(x-t)\,dt, \tag{1.24}$$

which was investigated in [23], [29], [30], [35], [36], [44], etc. It is well known ([12], Theorem 1.20) that the action of the operators R_j reduces in the range of the Fourier transformation to multiplication by $-ix_j/|x|$, so that

$$(\mathbf{R}, \mathbf{R}) = \sum_{j=1}^{n} R_j^2 = -I \tag{1.25}$$

and the operators R_j are bounded in $L_p(R^n)$, $1 < p < \infty$ (by a theorem of Mihlin [12] on multipliers). The Riesz transform relates the Poisson operator P_a to the Poisson adjoint operators $\widetilde{P}_{a,j}$. Namely, let

$$(P_a f)(x) = \frac{a}{c_n} \int_{R^n} \frac{f(x-t)\,dt}{(|t|^2+a^2)^{\frac{n+1}{2}}},$$

$$(\widetilde{P}_{a,j} f)(x) = \frac{1}{c_n} \int_{R^n} \frac{t_j f(x-t)\,dt}{(|t|^2+a^2)^{\frac{n+1}{2}}}, \tag{1.26}$$

where $a \in R^1$ and $j = 1, \ldots, n$; then (see G. O. Okikiolu [39]; [41], p. 485)

$$\widetilde{P}_{a,j} f = R_j P_a f = P_a R_j f, \quad f \in L_p(R^n), \quad 1 < p < \infty. \tag{1.27}$$

In §5 we use the following result.

LEMMA 5. *Let* $K(x) \in L_1(R^n) \cap L_r(R^n)$, $r > 1$, *and let*

$$K_N f = \frac{1}{N^n} \int_{R^n} \mathcal{K}\left(\frac{x-t}{N}\right) f(t)\,dt, \quad f(t) \in L_p(R^n).$$

Then $\|K_N f\|_p \to 0$ *as* $N \to \infty$, $1 < p < \infty$.

PROOF. In view of the uniform bound $\|K_N\|_p \leqslant \|K\|_1 \cdot \|f\|_p$, according to the Banach-Steinhaus theorem [5] it is sufficient to verify that $K_N f \xrightarrow[L_p]{} 0$ on a dense set in $L_p(R^n)$, for example, for $f \in C_0^\infty$. By means of simple transformations it is not difficult to arrive at the bound

$$\|K_N f\|_p \leqslant cN^{-\frac{n}{p'r'}} \|\mathcal{K}\|_1^{\frac{1}{p}} \cdot \|\mathcal{K}\|_r^{\frac{1}{p'}} \cdot \|f\|_{p/\nu},$$

where $1 < \nu < p$ and $c = c(\operatorname{supp} f; n, p, r, \nu)$ is independent of N.

§2. The spaces $L_{p,r}^\alpha$ and $I^\alpha(L_p)$. Integral representations

DEFINITION 1. We denote by $L_{p,r}^\alpha = L_{p,r}^\alpha(R^n)$, $1 < p < \infty$, $1 < r < \infty$, $0 < \alpha < \infty$, the class of functions $f(x)$ for which

$$f(x) \in L_r(R^n) \qquad \text{and} \qquad D_l^\alpha f \in L_p(R^n) \tag{2.1}$$

for arbitrary $l > 2[\alpha/2]$ ($l = \alpha$ for $\alpha = 1, 3, 5, \ldots$).

DEFINITION 2. Let $1 < p < n/\alpha$ and $0 < \alpha < n$. Denote by $I^\alpha(L_p)$ the class of functions $f(x)$ which can be represented by a Riesz potential $f(x) = (K^\alpha\varphi)(x)$ with density $\varphi \in L_p(R^n)$.

It is obvious that $I^\alpha(L_p) \subset L_{p_\alpha}$ by a theorem of Sobolev [22]. The fundamental result of §§2 and 3 will consist in the assertion that $L_{p,r}^\alpha = I^\alpha(L_p) \cap L_r$ in the case $1 < p < n/\alpha$, $p \leqslant r \leqslant p_\alpha$. Hence, in particular, it will follow that for the indicated values of p and r the definition of the space $L_{p,r}^\alpha$ is independent of the choice of $l > 2[\alpha/2]$.

LEMMA 6. *Let* $f(x) = (K^\alpha\varphi)(x)$, $0 < \alpha < n$, *and* $\varphi(x) \in L_p(R^n)$, $1 < p < n/\alpha$. *Then*

$$(\Delta_t^l f)(x) = |t|^\alpha \int_{R^n} k_{l,\alpha}(\xi)\, \varphi(x - |t| \operatorname{rot}_t \xi)\, d\xi, \tag{2.2}$$

where $l = 1, 2, \ldots$ *and*

$$k_{l,\alpha}(\xi) = \frac{1}{\gamma_n(\alpha)} \sum_{k=0}^{l} (-1)^k C_l^k |\xi - k\vec{j}|^{\alpha-n}. \tag{2.3}$$

In fact,

$$(\Delta_t^l f)(x) = \frac{1}{\gamma_n(\alpha)} \sum_{k=0}^{l} (-1)^k C_l^k \int_{R^n} \frac{\varphi(x-y)\, dy}{|y - kt|^{n-\alpha}}$$

$$= \frac{|t|^\alpha}{\gamma_n(\alpha)} \int_{R^n} \sum_{k=0}^{l} \frac{(-1)^k C_l^k}{|\xi - k\vec{j}|^{n-\alpha}} \varphi(x - |t| \operatorname{rot}_t \xi)\, d\xi,$$

which coincides with (2.2).

The following information relative to the kernel $k_{l,\alpha}(\xi)$ is essential for what follows.

LEMMA 7. *Suppose that* $l > 2[\alpha/2]$. *Then*

$$\int_{R^n} k_{l,\alpha}(\xi)\, d\xi = 0; \tag{2.4}$$

moreover, $k_{l,\alpha}(\xi) \in L_1(R^n)$ *in the case* $l > \alpha$, *and in the case* $2[\alpha/2] < l \leqslant \alpha$ *the integral in* (2.4) *is understood as conditionally convergent*:

$$\int_{R^n} k_{l,\alpha}(\xi)\, d\xi = \lim_{N\to\infty} \int_{\left|\xi - \frac{l}{2}\vec{j}\right| < N} k_{l,\alpha}(\xi)\, d\xi.$$

PROOF. 1) We show that

$$|k_{l,\alpha}(\xi)| \leqslant \frac{c}{|\xi|^{n+l-\alpha}} \quad \text{as} \quad \xi \to \infty, \tag{2.5}$$

which also ensures that $k_{l,\alpha}(\xi)$ belongs to the class $L_1(R^n)$ for $l > \alpha$. We use the notation

$$w(s) = \gamma_n^{-1}(\alpha)\,|\xi + s\vec{j}|^{\alpha-n}, \quad 0 < s < 1,$$

so that $k_{l,\alpha}(\xi) = (\Delta_1^l w)(0)$. By the well-known ([38]; [24], §3.3, formula (4)) identity

$$(\Delta_h^l w)(s) = \int_0^h \dots \int_0^h w^{(l)}(s + s_1 + \dots + s_l)\, ds_1 \dots ds_l, \quad h \in R^1,$$

we obtain that $k_{l,\alpha}(\xi) = w^{(l)}(s+\theta)|_{s=0}$, $0 < \theta < l$, so that

$$|k_{l,\alpha}(\xi)| \leqslant c\,|\xi + \theta\vec{j}|^{-n-l+\alpha},$$

as required.

2) For an odd positive integer l we have

$$\int_{\left|\xi - \frac{l}{2}\vec{j}\right| < N} k_{l,\alpha}(\xi)\, d\xi = 0, \tag{2.6}$$

for arbitrary $N > 0$. In fact,

$$\gamma_n(\alpha) \int_{\left|\xi - \frac{l}{2}\vec{j}\right| < N} k_{l,\alpha}(\xi)\, d\xi = \int_{\substack{|y|<N \\ y_1>0}} \sum_{k=0}^{l} (-1)^k C_l^k \left|y + \left(\frac{l}{2} - k\right)\vec{j}\right|^{\alpha-n} dy$$

$$+ \int_{\substack{|y|<N \\ y_1>0}} \sum_{k=0}^{l} (-1)^k C_l^k \left|y - \left(\frac{l}{2} - k\right)\vec{j}\right|^{\alpha-n} dy.$$

Replacing the index of summation k by $l - k$ in the second term in the right side, we obtain that it differs from the first by the factor $(-1)^l = -1$, which proves (2.6).

3) It remains to verify (2.4). For an odd l it is obvious in view of (2.6). Let l be even. Then $l > \alpha$ and $k_{l,\alpha}(\xi) \in L_1(R^n)$. In (2.2) we take $\varphi(x) \in \Phi$. Then also $f = K^\alpha \varphi \in \Phi$ (see [7]). We let $t \to 0$ in (2.2). Since $|(\Delta_t^l f)(x)| \leqslant c|t|^l$ for $f \in \Phi$ (compare with (1.21)), and $l > \alpha$, it is necessary to obtain that

$$\lim_{t\to 0}\int_{R^n} k_{l,\alpha}(\xi)\,\varphi(x-|t|\,\mathrm{rot}_t\xi)\,d\xi = 0.$$

Here, by a theorem of Lebesgue ([13], Chapter VI, §3, Theorem 1) we may pass to the limit under the integral sign, which yields (2.4). Lemma 7 is proved.

In the following theorem we obtain an integral representation for the truncated Riesz derivative $\mathbf{D}^{\alpha}_{l,\epsilon}f$. In what follows we shall not stipulate every time that $l=\alpha$ in the case of an odd positive integer α.

THEOREM 1. *Let $f(x)=K^{\alpha}\varphi$, $0<\alpha<n$, and $\varphi\in L_p(R^n)$, $1\leqslant p<n/\alpha$. Then*

$$(\mathbf{D}^{\alpha}_{l,\epsilon}f)(x)=\int_{R^n}\mathcal{K}_{l,\alpha}(|y|)\,\varphi(x-\epsilon y)\,dy,\quad l>2\left[\frac{\alpha}{2}\right], \tag{2.7}$$

where

$$\mathcal{K}_{l,\alpha}(|y|)=\frac{1}{d_{n,l}(\alpha)\,|y|^n}\int_{|\xi|<|y|}k_{l,\alpha}(\xi)\,d\xi\in L_1(R^n);$$

moreover,

$$\int_{R^n}\mathcal{K}_{l,\alpha}(|y|)\,dy=1. \tag{2.8}$$

PROOF. We have

$$(\mathbf{D}^{\alpha}_{l,\epsilon}f)(x)=\frac{1}{d_{n,l}(\alpha)\,\gamma_n(\alpha)}\int_{|t|>\epsilon}\frac{dt}{|t|^{n+\alpha}}\int_{R^n}\varphi(x-y)\sum_{k=0}^{l}\frac{(-1)^kC_l^k}{|y-kt|^{n-\alpha}}\,dy$$

$$=\frac{1}{d_{n,l}(\alpha)\,\gamma_n(\alpha)}\int_{R^n}\frac{\varphi(x-y)}{|y|^n}\,dy\int_{|t|>\frac{\epsilon}{|y|}}\sum_{k=0}^{l}(-1)^kC_l^k\frac{\left|\frac{y}{|y|}-kt\right|^{\alpha-n}}{|t|^{n+\alpha}}\,dt.$$

Hence after the substitution $t=\mathrm{rot}_y\eta$ and $y=\epsilon\xi$ (and then $\eta=\tau/|\tau|^2$) we obtain

$$(\mathbf{D}^{\alpha}_{l,\epsilon}f)(x)=\frac{1}{d_{n,l}(\alpha)\,\gamma_n(\alpha)}\int_{R^n}\frac{\varphi(x-\epsilon\xi)}{|\xi|^n}\,d\xi\int_{|\eta|>\frac{1}{|y|}}\frac{\sum_{k=0}^{l}(-1)^kC_l^k\,|\vec{j}-k\eta|^{\alpha-n}}{|\eta|^{n+\alpha}}\,d\eta \tag{2.9}$$

$$=\frac{1}{d_{n,l}(\alpha)\,\gamma_n(\alpha)}\int_{R^n}\frac{\varphi(x-\epsilon\xi)}{|\xi|^n}\,d\xi\int_{|\tau|<|y|}\sum_{k=0}^{l}(-1)^kC_l^k\left||\tau|\vec{j}-k\frac{\tau}{|\tau|}\right|^{\alpha-n}d\tau.$$

Since

$$\left||\tau|\vec{j}-k\frac{\tau}{|\tau|}\right|=\sqrt{|\tau|^2-2\tau_1k+k^2}=|\tau-k\vec{j}|, \tag{2.10}$$

(2.9) coincides with (2.7), and it remains to show that $K_{l,\alpha}(|y|)\in L_1(R^n)$. Since

$$|k_{l,\alpha}(t)|\leqslant c|t|^{\alpha-n}$$

as $|t| \to 0$, we have also $|K_{l,\alpha}(|y|)| \leqslant c_1 |y|^{\alpha-n}$ as $|y| \to 0$. We estimate $K_{l,\alpha}(|y|)$ as $|y| \to \infty$. If $l > \alpha$, then by (2.4) and (2.5)

$$|\mathscr{K}_{l,\alpha}(|y|)| = \frac{1}{|d_{n,l}(\alpha)| \cdot |y|^n} \left| \int\limits_{|t|>|y|} k_{l,\alpha}(t)\, dt \right| \leqslant \frac{c}{|y|^{n+l-\alpha}}. \tag{2.11}$$

Let $2[\alpha/2] < l \leqslant \alpha$, so that l is odd. In this case the summability of $K_{l,\alpha}(|y|)$ on R^n can be obtained at the expense, represented by equation (2.6), of integrating not over the sphere $|t| < |y|$, but over the layer $|y| - l/2 < |t| < |y| + l/2$. Namely, by (2.6) we have, for $|y| > l/2$,

$$\mathscr{K}_{l,\alpha}(|y|) = \frac{1}{|d_{n,l}(\alpha)|\,|y|^n} \left(\int\limits_{|t|<|y|} - \int\limits_{\left|t - \frac{l}{2}\vec{j}\right|<|y|} \right) k_{l,\alpha}(t)\, dt,$$

so that, taking account of (2.5),

$$\begin{aligned} |\mathscr{K}_{l,\alpha}(|y|)| &\leqslant \frac{1}{d_{n,l}(\alpha)|y|^n} \int\limits_{|y|-\frac{l}{2}<|t|<|y|+\frac{l}{2}} |k_{l,\alpha}(t)|\, dt \\ &\leqslant \frac{c}{|y|^n} \left| \left(|y| + \frac{l}{2}\right)^{\alpha-l} - \left(|y| - \frac{l}{2}\right)^{\alpha-l} \right| \leqslant \frac{c}{|y|^{n+l-\alpha+1}} \end{aligned} \tag{2.12}$$

as $|y| \to \infty$, which guarantees the condition $K_{l,\alpha}(|y|) \in L_1(R^n)$. Equation (2.8), which is a consequence of the choice of the normalizing constants $\gamma_n(\alpha)$ and $d_{n,l}(\alpha)$, can be established by indirect means: we choose $\varphi(x) \in \Phi$ so that $f(x) \in \Phi$ and in (2.7) it is possible to pass to the limit as $\epsilon \to 0$, which yields the equality

$$\mathbf{D}^\alpha f = \varphi(x) \int\limits_{R^n} \mathscr{K}_{l,\alpha}(|y|)\, dy.$$

Since $\mathbf{D}^\alpha f = \varphi$ (for $f, \varphi \in \Phi$ this is obviously at the expense of passing to the Fourier transforms), hence (2.8) follows. Theorem 1 is proved.

The following result follows immediately from Theorem 1.

THEOREM 2. *For any choice of $l > 2[\alpha/2]$ the operator $\mathbf{D}_l^\alpha$ is a left inverse of the Riesz operator K^α in $L_p(R^n)$:*

$$\mathbf{D}_l^\alpha K^\alpha \varphi \equiv \varphi, \quad \varphi \in L_p(R^n), \quad 1 \leqslant p < \frac{n}{\alpha}.$$

In fact, from (2.7), taking (2.8) into account, we have, applying the generalized Minkowski inequality,

$$\|\mathbf{D}_\varepsilon^\alpha f - \varphi\|_p \leqslant \int\limits_{R^n} |\mathscr{K}_{l,\alpha}(|y|)|\, \omega_p^1(\varphi, \varepsilon y)\, dy \to 0$$

by Lebesgue's theorem on passing to the limit under an integral sign.

The following theorem, which extends the representation (2.2) to the case where

the information on representability of $f(x)$ by a Riesz potential is replaced by the condition $f(x) \in L^{\alpha}_{p,r}$, also allows us below, in Theorem 4, to obtain a description of the space $I^{\alpha}(L_p)$.

THEOREM 3. *Let $f(x) \in L_p(R^n)$ and $\mathrm{D}^{\alpha}_l f \in L_p(R^n)$, where $1 < p < n/\alpha$, $p \leqslant r \leqslant p_{\alpha}$ and $l > 2[\alpha/2]$. Then for the difference $(\Delta^m_h f)(x)$ we have the representation* (2.2) *for $m > \alpha$, or, what is the same,*

$$(\Delta^m_h f)(x) = \int_{R^n} a_{m,h}(x-t)\,(\mathrm{D}^{\alpha} f)(t)\,dt, \tag{2.13}$$

where

$$a_{m,h}(x) = \frac{1}{\gamma_n(\alpha)} \sum_{k=0}^{m} (-1)^k C^k_m \,|x - kh|^{\alpha-n}, \tag{2.14}$$

so that

$$\|a_{m,h}\|_1 = c\,|h|^{\alpha} \quad \text{and} \quad \widehat{a}_{m,h}(x) = |x|^{-\alpha}\,|1 - e^{i(x,h)}|^m, \tag{2.15}$$

where $c = c(n, \alpha, m)$ is independent of h.

PROOF. Letting $\varphi_{\epsilon} = \mathrm{D}^{\alpha}_{\epsilon} f$ and $A\varphi = a_{m,h} * \varphi$, we have

$$\begin{aligned} A\varphi_{\epsilon} = \frac{1}{d_{n,l}(\alpha)} \Bigg\{ & \int_{R^n} a_{m,h}(x-t) f(t) \left(\int_{|t-y|>\epsilon} \frac{dy}{|t-y|^{n+\alpha}} \right) dt \\ & + \sum_{\nu=1}^{l} (-1)^{\nu} C^{\nu}_l \nu^{\alpha} \int_{R^n} a_{m,h}(x-t)\,dt \int_{|t-y|>\nu\epsilon} \frac{f(y)\,dy}{|t-y|^{n+\alpha}} \Bigg\}. \end{aligned} \tag{2.16}$$

Since $m > \alpha$, we have $a_{m,h}(x) \in L_1(R^n)$ (see the proof of the fact that $k_{l,\alpha}(t) \in L_1(R^n)$ for $l > \alpha$). Therefore we may change the order of integration in (2.16), so that

$$\begin{aligned} A\varphi_{\epsilon} &= \frac{1}{d_{n,l}(\alpha)} \sum_{\nu=0}^{l} (-1)^{\nu} C^{\nu}_l \int_{R^n} f(y)\,dy \int_{|\tau|>\epsilon} \frac{a_{m,h}(x-y-\nu\tau)}{|\tau|^{n+\alpha}}\,d\tau \\ &= \frac{1}{\gamma_n(\alpha)\,d_{n,l}(\alpha)} \int_{R^n} \sum_{k=0}^{m} (-1)^k C^k_m f(x-kh-y) \sum_{\nu=0}^{l} (-1)^{\nu} C^{\nu}_l\,dy \int_{|\tau|>\epsilon} \frac{|y-\nu\tau|^{\alpha-n}}{|\tau|^{n+\alpha}}\,d\tau; \end{aligned}$$

hence after the substitution $y \longrightarrow \epsilon y$, $\tau \longrightarrow \epsilon|y|\tau$ we find that

$$A\varphi_{\epsilon} = \frac{1}{\gamma_n(\alpha)\,d_{n,l}(\alpha)} \int_{R^n} \frac{(\Delta^m_h f)(x-\epsilon y)}{|y|^n}\,dy \int_{|\tau|>\frac{1}{|y|}} \frac{\sum_{\nu=0}^{l} (-1)^{\nu} C^{\nu}_l \left| \frac{y}{|y|} - \nu\tau \right|^{\alpha-n}}{|\tau|^{\alpha+n}}\,d\tau.$$

Setting $\tau = \mathrm{rot}_y(t/|t|^2)$, we obtain

$$A\varphi_{\epsilon} = \frac{1}{\gamma_n(\alpha)\,d_{n,l}(\alpha)} \int_{R^n} \frac{(\Delta^m_h f)(x-\epsilon y)}{|y|^n}\,dy \int_{|t|<|y|} \sum_{\nu=0}^{l} (-1)^{\nu} C^{\nu}_l \left| |t|\vec{j} - \nu \frac{t}{|t|} \right|^{\alpha-n} dt,$$

which, taking account of (2.10), reduces to

$$\int_{R^n} a_{m,h}(x-t)\,\varphi_\varepsilon(t)\,dt = \int_{R^n} (\Delta_h^m f)(x-\varepsilon y)\,\mathcal{K}_{l,\alpha}(|y|)\,dy. \tag{2.17}$$

Hence the equality (2.13) to be proved is obtained as $\epsilon \to 0$, taking account of (2.8). We justify the passage to the limit in (2.17), having carried it out in the metric of $L_r(R^n)$. In the left side this is possible since the operator A is bounded from $L_p(R^n)$ into $L_r(R^n)$ for $p \leqslant r \leqslant p_\alpha$ (for $p \leqslant r < p_\alpha$ boundedness of A follows from a theorem of Young on convolutions, since $a_{m,h}(x) \in L_q(R^n)$ for $1 \leqslant q < n/(n-\alpha)$; and for $r = p_\alpha$ it follows from a theorem of Sobolev). For the right side we have, applying a generalized Minkowski inequality and taking account of (2.8),

$$\left\| \int_{R^n} \mathcal{K}_{l,\alpha}(|y|)\,(\Delta_h^m f)(x-\varepsilon y)\,dy - (\Delta_h^m f)(x) \right\|_r \leqslant \int_{R^n} \mathcal{K}_{l,\alpha}(|y|)\,w_r^1(\Delta_h^m f, \varepsilon y)\,dy \to 0$$

by a theorem of Lebesgue on passing to the limit, since $\Delta_h^m f \in L_r(R^n)$ and $\mathcal{K}_{l,\alpha}(|y|) \in L_1(R^n)$.

The second of the equations (2.15) is a consequence of the formula $\widehat{\tau_h f}(x) = e^{i(x,h)}\hat{f}(x)$, and the first we obtain from the equality

$$\|a_{m,h}\|_1 = |h|^\alpha \int_{R^n} \left| \sum_{k=0}^m (-1)^k C_m^k \left| x - k\frac{h}{|h|} \right| \right|^{\alpha-n} dx,$$

resorting to the rotation $x = \mathrm{rot}_h t$. And finally, the agreement of (2.13) with (2.2) follows from

$$a_{m,h}(x) = \frac{|h|^{\alpha-n}}{\gamma_n(\alpha)} \sum_{=0}^m (-1)^k C_m^k \left| \mathrm{rot}_h^{-1}\frac{x}{|h|} - k\,\mathrm{rot}_h^{-1}\frac{h}{|h|} \right|^{\alpha-n} = |h|^{\alpha-n} k_{m,\alpha}\left(\mathrm{rot}_h^{-1}\frac{x}{|h|}\right).$$

Theorem 3 is proved.

§3. The spaces $I^\alpha(L_p)$ and $L^\alpha_{p,r}$ (continued)

The following result is the fundamental assertion of this section.

THEOREM 4 (a description of the space $I^\alpha(L_p)$). *In order that $f(x) \in I^\alpha(L_p)$ it is necessary and sufficient that*

$$f(x) \in L_{p_\alpha}(R^n), \quad \mathbf{D}_l^\alpha f \in L_p(R^n), \quad l > 2\left[\frac{\alpha}{2}\right]; \tag{3.1}$$

in addition,

$$I^\alpha(L_p) \cap L_r = L^\alpha_{p,r} \tag{3.2}$$

for all $p \leqslant r \leqslant p_\alpha$.

PROOF. Necessity of the condition $f(x) \in L_{p_\alpha}$ follows from a theorem of Sobolev [22]; the condition $\mathbf{D}_l^\alpha f \in L_p$, from Theorem 2. Consequently, $I^\alpha(L_p) \cap L_r \subseteq L_{p,r}^\alpha$. It remains to prove the reverse inclusion. Let $f(x) \in L_{p,r}^\alpha$. According to Theorem 3 we have the representation (2.13). Noting that the right side in (2.13) is $(\Delta_h^m K^\alpha \mathbf{D}^\alpha f)(x)$, we obtain

$$\Delta_h^m f \equiv \Delta_h^m K^\alpha \mathbf{D}^\alpha f. \tag{3.3}$$

It is not difficult to see that functions which have identically coinciding finite differences may differ only by a polynomial (for this it is sufficient to pass to the Fourier transform in S':

$$[1 - e^{i(x,h)}]^m \cdot [\hat{f}(x) - \widehat{K^\alpha \mathbf{D}^\alpha f}(x)] \equiv 0, \quad x \in R^n, \quad h \in R^n,$$

and to use a theorem on the general form of a functional which is concentrated at a point—see [3], Chapter II, §4.5). Since the functions $f \in L_r(R^n)$ and $K^\alpha \mathbf{D}^\alpha f \in L_{p_\alpha}(R^n)$ do not "contain" polynomials, we have

$$f = K^\alpha \mathbf{D}^\alpha f \in I^\alpha(L_p), \tag{3.4}$$

which proves Theorem 4. We note that in the case of a space of Bessel potentials ($r = p$) the equality $I^\alpha(L_p) \cap L_p = L_{p,p}^\alpha$ follows from results of Lizorkin [8]–[10].

COROLLARY 1. $C_0^\infty \subset I^\alpha(L_p)$.

In fact, if $\alpha \neq 1, 3, 5, \ldots$, then it is sufficient to verify that $\mathbf{D}_l^\alpha f \in L_p$ for $f \in C_0^\infty$ at least for $l > \alpha$. We have

$$\|\mathbf{D}_l^\alpha f\|_p \leqslant c_1 \|f\|_p \int_{|t|>1} |t|^{-n-\alpha}\, dt + c_2 \int_{|t|<1} |t|^{-n-\alpha} \omega_p(f, t)\, dt < \infty$$

by (1.22). If $\alpha = 1, 3, 5, \ldots$, then $l = \alpha$, and a similar bound can be obtained with the help of (1.4).

COROLLARY 2. *If $f \in L_r(R^n)$ and $\mathbf{D}^\alpha f \in L_p(R^n)$, $1 < p < n/\alpha$, $p \leqslant r \leqslant p_\alpha$, then $\mathbf{D}^{\alpha-\epsilon} f \in L_{p_\epsilon}(R^n)$, $0 < \epsilon < \alpha$, and*

$$\|\mathbf{D}^{\alpha-\epsilon} f\|_{p_\epsilon} \leqslant c \|\mathbf{D}^\alpha f\|_p. \tag{3.5}$$

We introduce a norm in $L_{p,r}^\alpha(R^n)$ by the equation

$$\|f\|_{L_{p,r}^\alpha(R^n)} = \|f\|_r + \|\mathbf{D}^\alpha f\|_p. \tag{3.6}$$

THEOREM 5 (On imbedding of the spaces $L_{p,r}^\alpha$ with respect to the parameter r).

$$L_{p,r_1}^\alpha \to L_{p,r_2}^\alpha, \tag{3.7}$$

if $1 < p < n/\alpha$ and $p \leqslant r_1 \leqslant r_2 \leqslant p_\alpha$.

The theorem is a consequence of (3.2) and the interpolational inequality

$$\|f\|_{r_2} \leqslant \|f\|_{r_1}^{\mu} \cdot \|f\|_{p_\alpha}^{1-\mu}, \quad \mu = \frac{r_1(p_\alpha - r_2)}{r_2(p_\alpha - r_1)}.$$

In fact, for $f \in L^{\alpha}_{p,r}$, according to (3.4) we have $\|f\|_{p_\alpha} \leqslant c\|\mathbf{D}^{\alpha} f\|_p$ by a theorem of Sobolev, and then

$$\|f\|_{L^{\alpha}_{p,r_2}} \leqslant \|\mathbf{D}^{\alpha} f\|_p^{1-\mu} (\|f\|_{r_1}^{\mu} + \|\mathbf{D}^{\alpha} f\|_p^{\mu})$$

$$\leqslant 2^{1-\mu} c \|\mathbf{D} f\|_p^{1-\mu} (\|f\|_{r_1} + \|\mathbf{D}^{\alpha} f\|_p)^{\mu} \leqslant c_1 \|f\|_{L^{\alpha}_{p,r_1}}.$$

THEOREM 6 (on imbedding of the spaces $L^{\alpha}_{p,r}$ with respect to the parameter α).

$$L^{\alpha}_{p,r} \to L^{\alpha-\varepsilon}_{p_\varepsilon,r}, \tag{3.8}$$

if $1 < p < n/\alpha$, $p \leqslant r \leqslant p_\alpha$ *and* $0 < \epsilon < \alpha$; *in particular,*

$$I^{\alpha}(L_p) \to I^{\beta}(L_{p_{\alpha-\beta}}), \quad 0 < \beta < \alpha. \tag{3.9}$$

PROOF. Let $f(x) \in L^{\alpha}_{p,r}$. By (3.4), $f = K^{\alpha-\epsilon} K^{\epsilon} \mathbf{D}^{\alpha} f$. Hence, according to Theorem 2 we have $\mathbf{D}^{\alpha-\epsilon} f = K^{\epsilon} \mathbf{D}^{\alpha} f$, so that by the Sobolev theorem

$$\|\mathbf{D}^{\alpha-\varepsilon} f\|_{p_\varepsilon} \leqslant c\|\mathbf{D}^{\alpha} f\|_p.$$

But then

$$\|f\|_{L^{\alpha-\varepsilon}_{p_\varepsilon,r}} \leqslant c\|f\|_{L^{\alpha}_{p,r}}.$$

COROLLARY. $L^{\alpha}_{p,r} \to L^{\alpha-\epsilon}_{p_\epsilon,r+\delta}$, *if* $1 < p < n/\alpha$, $p \leqslant r \leqslant p_\alpha$, $0 < \epsilon < \alpha$ *and* $0 < \delta < p_\alpha - r$.

We also note certain properties of Riesz potentials which are related to the modulus of continuity. Let $f(x) \in L^{\alpha}_{p,r}$, $1 < p < n/\alpha$, $p \leqslant r \leqslant p_\alpha$ and $l > \alpha$ ($\alpha \neq 1, 3, \ldots$). Then

$$1) \quad \omega^{l}_{p}(f, h) \leqslant c\,|h|^{\alpha} \|\mathbf{D}^{\alpha} f\|_p, \tag{3.10}$$

$$2) \quad \omega^{l}_{p}(f, h) = o(|h|^{\alpha}) \quad \text{as} \quad h \to 0. \tag{3.11}$$

Hence (3.10) follows from (2.2) in view of the fact that $k_{l,\alpha}(t) \in L_1(R^n)$ for $l > \alpha$. The estimate (3.11), which is due to Hardy and Littlewood [33] in the one-dimensional periodic case, also follows from (2.2), in view of property (2.4) of the kernel $k_{l,\alpha}(t)$. (Concerning the one-dimensional nonperiodic case, see also [17] and [20].)

In concluding this section we note that (as follows from Theorem 4) in the description $I^{\alpha}(L_p) = \{f: f \in L_{p_\alpha}, \mathbf{D}^{\alpha} f \in L_p, \omega^{1}_{p}(f, h) = o(|h|^{\alpha})\}$ of the space $I^{\alpha}(L_p)$, which was obtained in [18] and [20] in the one-dimensional case, the last condition is a consequence of the first two.

§4. On weak derivatives of integral order in $I^{\alpha}(L_p)$

In this section we shall show: 1) the existence of weak derivatives $D^{\beta} f$ of integral order $|\beta| \leqslant \alpha$ for functions $f(x) \in L_r$ which have a Riesz derivative $\mathbf{D}^{\alpha} f \in L_p$ (Theorem 7); 2) the validity of (1.19) for such functions (Theorem 8). A weak derivative will be understood in the sense of generalized functions over Φ. Namely, we shall say that a locally summable function $f(x) \in S'$ has a weak (generalized) derivative $D^{\beta} f$ if there exists a locally summable function $g(x) \in S'$ such that

$$(g, \omega) = (-1)^{|\beta|} (f, D^{\beta}\omega), \quad \omega \in \Phi, \tag{4.1}$$

and then $D^{\beta}f \stackrel{\text{def}}{=} g$. A certain inconvenience of definition (4.1), which consists in the fact that it determines the function $g(x)$ "to within an additive polynomial" (Lizorkin, [8]), is not crucial here.

For $\alpha > |\beta|$ we introduce the ("quasi-Riesz") potential

$$K_{\beta}^{\alpha-|\beta|}\varphi = \frac{1}{\gamma_n(\alpha)} \int_{R^n} \varphi(x-t)\, D^{\beta}(|t|^{\alpha-n})\, dt, \tag{4.2}$$

which is in a sense similar to the Riesz potential $K^{\alpha-|\beta|}\varphi$, since $D^{\beta}(|t|^{\alpha-n}) = c(t)|t|^{\alpha-|\beta|-n}$, where $c(t)$ is a bounded function. The operator (4.2) can be represented as the composite of a Riesz potential with a Riesz transform:

$$K_{\beta}^{\alpha-|\beta|}\varphi = R^{\beta}K^{\alpha-|\beta|}\varphi, \tag{4.3}$$

where $\varphi \in L_p$, $1 < p < n/(\alpha - |\beta|)$, and

$$R^{\beta} = R_1^{\beta_1} \ldots R_n^{\beta_n}; \tag{4.4}$$

the R_j are the operators (1.24). In fact, since

$$K_{\beta}^{\alpha-|\beta|}\omega = (-1)^{|\beta|}D^{\beta}K^{\alpha}\omega = (-1)^{|\beta|}K^{\alpha}D^{\beta}\omega \tag{4.5}$$

(at least for $\omega \in \Phi$), (4.3) is obvious for $\varphi \in \Phi$ (pass to the Fourier transform). The validity of (4.3) for $\varphi \in L_p$ follows from the boundedness of the operators in (4.3) and the fact that Φ is dense in the L_p-spaces, which was established by Lizorkin in [8] (see also [10], Appendix 1).

THEOREM 7. *Functions $f(x) \in I^{\alpha}(L_p)$ have all weak derivatives $D^{\beta}f$ of integral order $|\beta| \leqslant \alpha$; moreover,*

$$\|D^{\beta}f\|_{p_{\alpha-|\beta|}} \leqslant c_1 \|\mathbf{D}^{|\beta|}f\|_{p_{\alpha-|\beta|}} \leqslant c_2 \|\mathbf{D}^{\alpha}f\|_p. \tag{4.6}$$

In addition, the derivatives $D^{\beta}f$ can be expressed in terms of the Riesz derivatives by the formula

$$D^{\beta}f = R^{\beta}\mathbf{D}^{|\beta|}f = K_{\beta}^{\alpha-|\beta|}\mathbf{D}^{\alpha}f.^{(1)} \tag{4.7}$$

PROOF. The existence of $\mathbf{D}^{|\beta|}f \in L_{p_{\alpha-|\beta|}}$ and the second of the bounds (4.6) are established in Corollary 2 of Theorem 4. The second inequality in (4.7) follows from (4.3) and the equation $K^{\alpha-|\beta|}\mathbf{D}^{\alpha}f = \mathbf{D}^{|\beta|}f$, $f \in I^{\alpha}(L_p)$. For a proof of the remaining equality in (4.7) it remains to show in accordance with (4.1) that $(K_{\beta}^{\alpha-|\beta|}\mathbf{D}^{\alpha}f, \omega) = (f, (-D)^{\beta}\omega)$. For $|\beta| < \alpha$ this follows from the following chain of equalities:

$$\begin{aligned}(K_{\beta}^{\alpha-|\beta|}\mathbf{D}^{\alpha}f, \omega) &= (\mathbf{D}^{\alpha}f, K_{\beta}^{\alpha-|\beta|}\omega) = (-1)^{|\beta|}(\mathbf{D}^{\alpha}f, K^{\alpha}D^{\beta}\omega)\\ &= (-1)^{|\beta|}(K^{\alpha}\mathbf{D}^{\alpha}f, D^{\beta}\omega) = (-1)^{|\beta|}(f, D^{\beta}\omega),\end{aligned}$$

(1) In accordance with (4.3), we assume that $K_{\beta}^{0} = R^{\beta}$ in the case $|\beta| = \alpha$ (= 1, 2, 3, ...).

in which the first and the third are justified by Fubini's theorem, the second follows from (4.5), and the fourth from (3.4).

For $|\beta| = \alpha$ it is necessary to justify the equality

$$(R^\beta\varphi, \omega) = (-1)^{|\beta|}(K^\alpha\varphi, D^\beta\omega), \quad \omega \in \Phi, \tag{4.8}$$

where $\varphi = \mathrm{D}^\alpha f \in L_p$. First we verify that

$$(R^\beta\varphi, \omega) = (-1)^{|\beta|}(\varphi, R^\beta\omega), \quad \varphi \in L_p, \quad 1 < p < \infty, \quad \omega \in \Phi. \tag{4.9}$$

In fact, since

$$|(R^\beta\varphi, \omega)| \leqslant \|R^\beta\|_{L_p \to L_p} \cdot \|\varphi\|_p \cdot \|\omega\|_{p'}$$

and

$$|(\varphi, R^\beta\omega)| \leqslant \|R^\beta\|_{L_{p'} \to L_{p'}} \|\varphi\|_p \cdot \|\omega\|_{p'},$$

the left and right sides in (4.9) are bounded functionals in $L_p(R^n)$, and therefore it is sufficient to verify that they agree on the dense set Φ. Since $R_j^\beta\varphi \in \Phi$ for $\varphi \in \Phi$, by (4.4) it is sufficient to verify that

$$(R_j\varphi, \omega) = -(\varphi, R_j\omega), \quad \varphi, \omega \in \Phi. \tag{4.10}$$

We put

$$R_{j,\varepsilon}\varphi = \frac{1}{c_n} \int\limits_{|x-t|>\varepsilon} \varphi(t) \frac{x_j - t_j}{|x-t|^{n+1}}\, dt.$$

In view of Fubini's theorem and the formula

$$\int\limits_{R^n} dx \int\limits_{|x-y|>\varepsilon} f(x, y)\, dy = \int\limits_{R^n} dy \int\limits_{|x-y|>\varepsilon} f(x, y)\, dx$$

we obtain

$$(R_{j,\varepsilon}\varphi, \omega) = -(\varphi, R_{j,\varepsilon}\omega).$$

But then

$$(R_j\varphi, \omega) = \lim_{\substack{\varepsilon \to 0 \\ (L_2)}} (R_{j,\varepsilon}\varphi, \omega) = -\lim_{\substack{\varepsilon \to 0 \\ (L_2)}} (\varphi, R_{j,\varepsilon}\omega) = -(\varphi, R_j\omega),$$

which proves (4.10), and together with this also (4.9). Equality (4.8) follows from (4.9) if we write it in the form $(R^\beta\varphi, \omega) = (-1)^{|\beta|}(\varphi, K^\alpha D^\beta\omega)$ by Fubini's theorem and consider that $K^\alpha D^\beta\omega = R^\beta\omega$ for $\omega \in \Phi$ (pass to the Fourier transform). And, finally, (4.6) follows immediately from (4.7). Theorem 7 is proved.

Turning to the question of replacement of even order Riesz derivatives $\mathrm{D}^{2m}f$ by a power of the Laplacian $(-\Delta)^m f$ applied to functions $f(x) \in I^\alpha(L_p)$, we formulate the following lemma, which is essentially a consequence of the equalities

$$(-\Delta)^m K^\alpha\omega = K^\alpha(-\Delta)^m\omega = K^{\alpha-2m}\omega, \quad 2m \leqslant \alpha,$$

which are well known for sufficiently "good" functions ω, for example, $\omega \in \Phi$. Everywhere below $(-\Delta)^m$ is understood in the weak sense.

LEMMA 8. *Let* $2m \leqslant \alpha < n/p$, $m = 0, 1, \ldots$. *Then*

$$(-\Delta)^m K^\alpha \varphi = K^{\alpha-2m}\varphi \tag{4.11}$$

for $\varphi \in L_p$, *and*

$$K^\alpha (-\Delta)^m f = K^{\alpha-2m} f, \tag{4.12}$$

if $(-\Delta)^m f \in L_p$ *and* $f \in L_r$, $1 < r < n/(\alpha - 2m)$.

From this we obtain the following result.

THEOREM 8. *If* $f(x) \in I^\alpha(L_p)$, $1 < p < n/\alpha$, $\alpha \geqslant 2m$, *then*

$$\mathbf{D}^\alpha f = \mathbf{D}^{\alpha-2m}(-\Delta)^m f = (-\Delta)^m \mathbf{D}^{\alpha-2m} f. \tag{4.13}$$

PROOF. Let $f(x) = K^\alpha \varphi \in I^\alpha(L_p)$. Then it follows from (4.11) and Theorem 2 that $\mathbf{D}^{\alpha-2m}(-\Delta)^m K^\alpha \varphi = \varphi$. Comparing this with the fact that $\mathbf{D}^\alpha f = \varphi$ also by Theorem 2, we obtain the first of equations (4.13). The second can also be proved using (4.11):

$$(-\Delta)^m \mathbf{D}^{\alpha-2m} f = (-\Delta)^m \mathbf{D}^{\alpha-2m} K^{\alpha-2m} K^{2m}\varphi = (-\Delta)^m K^{2m}\varphi = \varphi.$$

The theorem is proved.

COROLLARY. *The odd order Riesz derivatives* $\mathbf{D}^{2m+1} f$, $m = 0, 1, \ldots, 2m + 1 \leqslant \alpha$, *of a function* $f \in I^\alpha(L_p)$ *can be expressed in terms of weak derivatives by the formula*

$$\mathbf{D}^{2m+1} f = (-1)^{m+1} \sum_{j=1}^{n} R_j \frac{\partial}{\partial x_j} \Delta^m f. \tag{4.14}$$

In fact, since $\partial f/\partial x_j = R_j \mathbf{D}^1 f$ according to (4.7), we have

$$\mathbf{D}^1 f = -\sum_{j=1}^{n} R_j \frac{\partial}{\partial x_j} f$$

by (1.25). But then (4.14) follows from (4.13).

In concluding this section we note that it is not difficult to show that the description (3.1) of the space $I^\alpha(L_p)$ remains equivalent if we replace $\mathbf{D}^\alpha f$ by $\mathbf{D}^{\alpha-2m}(-\Delta)^m f$ or $(-\Delta)^m \mathbf{D}^{\alpha-2m} f$ in (3.1).

§5. Denseness of C_0^∞ in $L_{p,r}^\alpha(R^n)$

THEOREM 9. *The space* C_0^∞ *is dense in* $L_{p,r}^\alpha(R^n)$ *for* $1 < p < n/\alpha$ *and* $p \leqslant r \leqslant p_\alpha$.

For a proof of this theorem we shall employ the usual means–approximation of a function by its average and a "reduction". And while the approximation of $f(x) \in L_{p,r}^\alpha(R^n)$ by the average can be accomplished easily by the fact that both the mean and the norm (3.6) are translation invariant, the approximation by the "reduction", which is not translation invariant, encounters significant difficulties, which are related to the use of conditional convergence, reinforcing the necessity to work in spaces L_p and L_r with distinct values p and r (in particular, in the limit case $r = p_\alpha$). This difficulty can be overcome by proving denseness via induction (Lemma 9), which we base on the imbedding (3.9), having considered the case of odd integral α beforehand.

First of all we show that for a function $f(x) \in L^{\alpha}_{p,r}(R^n)$ its average

$$f_\delta(x) = \int_{R^n} a(t) f(x-\delta t)\, dt \in C^\infty \cap L^{\alpha}_{p,r}(R^n),$$

where the function $a(t) \in C_0^\infty$ is normalized by the condition $\int_{R^n} a(t)\, dt = 1$, converges to $f(x)$ in the norm of $L^{\alpha}_{p,r}(R^n)$. We have

$$\|f - f_\delta\|_{L^{\alpha}_{p,r}} = \|f - f_\delta\|_r + \|D^\alpha (f - f_\delta)\|_p$$

and we need consider only the second term. Since $D^\alpha_\epsilon f_\delta = (D^\alpha_\epsilon f)_\delta$ by Fubini's theorem, we have

$$D^\alpha f_\delta = \lim_{\substack{\epsilon \to 0 \\ (L_p)}} (D^\alpha_\epsilon f)_\delta = (D^\alpha f)_\delta$$

in view of boundedness of the averaging operator in $L_p(R^n)$. Consequently, the average commutes with the Riesz differential on functions from $L^{\alpha}_{p,r}(R^n)$: $D^\alpha f_\delta = (D^\alpha f)_\delta$, and therefore

$$\|D^\alpha (f - f_\delta)\|_p = \|D^\alpha f - (D^\alpha f)_\delta\|_p \to 0.$$

It remains to approximate $f(x)$ by a function with compact support.

Let $\mu(x)$ be an arbitrary function from C_0^∞ with carrier in the ball $|x| < 2$, which is identically one for $|x| \leqslant 1$ and such that $|\mu(x)| \leqslant 1$. It is necessary to show that the "reduction"

$$\mu_N(x) f(x) \stackrel{\text{def}}{=} \mu\left(\frac{x}{N}\right) f(x) \tag{5.1}$$

approximates $f(x)$ in $L^{\alpha}_{p,r}(R^n)$ as $N \to \infty$. We put

$$\nu(x) = 1 - \mu(x), \quad \nu_N(x) = \nu\left(\frac{x}{N}\right), \tag{5.2}$$

so that it is necessary to show that $\|\nu_N f\|_{L^{\alpha}_{p,r}} \to 0$ as $N \to \infty$, if $f \in L^{\alpha}_{p,r}(R^n)$. For this it is sufficient to show that

$$\|D^\alpha (\nu_N f)\|_p \xrightarrow[N \to \infty]{} 0, \quad f \in I^\alpha(L_p), \tag{5.3}$$

which constitutes the basis of the difficulty.

1. *Justification of* (5.3) *for* $0 < \alpha < 1$. Taking $D^\alpha = D^\alpha_1$ for $0 < \alpha < 1$, we have

$$D^\alpha (\nu_N f) = \nu_N D^\alpha f + \frac{1}{d_{n,1}(\alpha)} B_N f, \tag{5.4}$$

where

$$B_N f = \int_{R^n} \frac{\nu_N(x) - \nu_N(x-t)}{|t|^{n+\alpha}} f(x-t)\, dt. \tag{5.5}$$

It is obvious that $\nu_N D^\alpha f \xrightarrow[L_p]{} 0$. We show that

$$B_N f \underset{L_p}{\to} 0 \tag{5.6}$$

for all $f \in L_{p_\alpha}$ $(\supset I^\alpha(L_p))$. By (1.21) we have

$$|B_N f| \leqslant \frac{c}{N+|x|} \int_{R^n} \frac{|f(x-t)|\,dt}{|t|^{n+\alpha-1}\left(1+\frac{|x-t|}{N}\right)}, \tag{5.7}$$

and then

$$\begin{aligned} \|B_N f\|_p &\leqslant \frac{c}{N} \int_{R^n} \frac{dt}{|t|^{n+\alpha-1}} \left\{ \int_{R^n} \frac{|f(x-t)|^p dx}{\left(1+\frac{|x|}{N}\right)^p \left(1+\frac{|x-t|}{N}\right)^p} \right\}^{\frac{1}{p}} \\ &\leqslant \frac{c\|f\|_{p_\alpha}}{N} \int_{R^n} \frac{dt}{|t|^{n+\alpha-1}} \left\{ \int_{R^n} \frac{dx}{\left(1+\frac{|x|}{N}\right)^{\frac{n p}{\alpha r - p}} \left(1+\frac{|x-t|}{N}\right)^{\frac{n}{q}}} \right\}^{\frac{\alpha}{n} - \frac{1}{p} + \frac{1}{r}}, \end{aligned} \tag{5.8}$$

which after the substitution $t \to Nt$, $x \to Nx$ and an application of Lemma 4 yields the uniform bound

$$\|B_N f\|_p \leqslant c\|f\|_{p_\alpha} \int_{R^n} \frac{dt}{|t|^{n+\alpha-1}(1+|t|)^{\frac{2\mu-n}{\mu}}} = c_1 \|f\|_{p_\alpha}, \tag{5.9}$$

where c_1 is independent of N. Therefore by the Banach-Steinhaus theorem it is sufficient to verify (5.6) on a dense set in L_{p_α}. Assuming therefore that $f \in C_0^\infty$ and its support lies in the ball $|x| < a$, we obtain

$$\|B_N f\|_p \leqslant \frac{c}{N} \int_{R^n} \frac{dt}{|t|^{n+\alpha-1}} \left\{ \int_{|x|<a} \frac{dx}{\left(1+\frac{|x|}{N}\right)^p \left(1+\frac{|x-t|}{N}\right)^p} \right\}^{\frac{1}{p}} \tag{5.10}$$

$$\leqslant \frac{c_1}{N} \int_{|t|<2a} \frac{dt}{|t|^{n+\alpha-1}} + \frac{c}{N^{\alpha-\frac{n}{p}}} \int_{|t|>\frac{2a}{N}} \frac{dt}{|t|^{n+\alpha-1}} \left\{ \int_{|x|<\frac{a}{N}} \frac{dx}{(1+|x|)^p (1+|x-t|)^p} \right\}^{\frac{1}{p}}.$$

Since $|x-t| > |t| - a/N$ for $|x| < a/N$ and $|t| > 2a/N$, we have from (5.10)

$$\|B_N f\|_p \leqslant \frac{c_2}{N} + \frac{c_3}{N^\alpha} \int_{R^n} \frac{dt}{|t|^{n+\alpha-1}\left(1+|t|-\frac{a}{N_0}\right)} = \frac{c_2}{N} + \frac{c_4}{N^\alpha} \to 0,$$

where $N > N_0 > a$, as required.

2. *Extension of a theorem on denseness to the case* $\alpha > 1$ $(\alpha \neq 1, 3, 5, \ldots)$. The validity of the theorem on denseness of C_0^∞ in $I^\alpha(L_p)$ can be obtained for $0 < \alpha < 1$ by extension of the method of induction to the general case.

LEMMA 9. *Let $m = 1, 2, \dots, n-1$. If the theorem on denseness of C_0^∞ in $I^\alpha(L_p)$ holds for $0 < \alpha < m$, then it holds also for $m \leqslant \alpha < m+1$ $(\alpha \neq 1, 3, 5, \dots)$.*(2)

PROOF. It is necessary to show that

$$\|\mathbf{D}^\alpha(v_N f)\|_p \xrightarrow[N\to\infty]{} 0, \quad m \leqslant \alpha < m+1, \tag{5.11}$$

for $f \in I^\alpha(L_p)$ under the assumption that C_0^∞ is dense in $I^\gamma(L_q)$ for $\gamma < m$. By the formula

$$(\Delta_t^l af)(x) = \sum_{k=0}^{l} C_l^k (\Delta_t^k a)(x)(\Delta_t^{l-k} f)(x-kt)$$

we have

$$\mathbf{D}^\alpha(v_N f) = v_N \mathbf{D}^\alpha f + \frac{1}{d_{n,l}(\alpha)} \sum_{k=1}^{l} C_l^k B_{N,k} f, \tag{5.12}$$

where

$$(B_{N,k}f)(x) = \int_{R^n} \frac{(\Delta_t^k v_N)(x)(\Delta_t^{l-k} f)(x-kt)}{|t|^{n+\alpha}}\, dt, \quad k = 1, 2, \dots, l,$$

so that in the proof we need

$$\|B_{N,k} f\|_p \xrightarrow[N\to\infty]{} 0, \quad f \in I^\alpha(L_p), \quad k = 1, \dots, l. \tag{5.13}$$

The idea of the proof of (5.13) consists in the following. We shall prove the following assertion, which is stronger than (5.13):

$$\|B_{N,k} f\|_p \xrightarrow[N\to\infty]{} 0, \quad f \in I^{\gamma_k}(L_{q_k}), \quad q_k = p_{\alpha-\gamma_k}, \tag{5.14}$$

where $0 < \gamma_k < m$, $k = 1, 2, \dots, l$. From the feasibility of (5.14) for $\gamma_k < m < \alpha$ will follow, by the imbedding (3.9), the feasibility of (5.13). For a proof of (5.14) we first obtain the uniform bound

$$\|B_{N,k} f\|_p \leqslant c \|f\|_{I_{\gamma_k}(L_{q_k})}, \quad f \in I^{\gamma_k}(L_{q_k}) \tag{5.15}$$

for a certain choice of $\gamma_k \in (0, m)$, $k = 1, 2, \dots, l$; the constant c is independent of N. After this, by the Banach-Steinhaus theorem, (5.14) remains to be verified on a dense set in $I^{\gamma_k}(L_{q_k})$. Since $\gamma_k < m$, we have (by assumption!) that C_0^∞ serves as such a set. Verification of (5.14) for $f \in C_0^\infty$ concludes the proof of Lemma 9.

We are using the possibility (provided by Theorem 4) of assuming that $l > \alpha$ in subsequent estimates. (The choice $l > \alpha$ is essential; the case $\alpha = 1, 3, 5, \dots$, which forces the choice $l = \alpha$, is considered separately.)

1) We obtain the uniform bound (5.15). Using Lemma 3, we have

$$|B_{N,k} f| \leqslant \frac{c}{N^k} \int_{R^n} \frac{|(\Delta_t^{l-k} f)(x-kt)|\, dt}{|t|^{n+\alpha-k} \left(1 + \frac{|x|}{N}\right)\left(1 + \frac{|x-t|}{N}\right) \dots \left(1 + \frac{|x-kt|}{N}\right)}$$

and then

(2) The case $\alpha = m$ for $m = 1, 3, 5, \dots$ is excluded from consideration.

$$\|B_{N,k}f\| \leqslant \frac{c}{N^k} \int\limits_{R^n} \frac{dt}{|t|^{n+\alpha-k}} \left\{ \int\limits_{R^n} \frac{|(\Delta_t^{l-k} f)(x-kt)|^p dx}{\left(1+\frac{|x|}{N}\right)^p \cdots \left(1+\frac{|x-kt|}{N}\right)^p} \right\}^{\frac{1}{p}}. \tag{5.16}$$

If $k = l$, then

$$\|B_{N,l}f\|_p \leqslant \frac{c\|f\|_{p_\alpha}}{N^l} \int\limits_{R^n} \frac{dt}{|t|^{n+\alpha-l}} \left\{ \int\limits_{R^n} \frac{dx}{\left(1+\frac{|x|}{N}\right)^{\frac{n}{\alpha}} \cdots \left(1+\frac{|x-lt|}{N}\right)^{\frac{n}{\alpha}}} \right\}^{\frac{\alpha}{n}}$$

and after the substitution $x \to Nx$, $t \to Nt$ and application of Lemma 4 we obtain

$$\|B_{N,l}f\|_p \leqslant c_1 \|f\|_{p_\alpha} \int\limits_{R^n} \frac{dt}{|t|^{n+\alpha-l}(1+|t|)^{l-\alpha+1-\varepsilon}} = c_2 \|f\|_{p_\alpha}; \tag{5.17}$$

if $k = 1, 2, \ldots, l-1$, then, applying the Hölder inequality with exponents $p_1 = q_k/p$ and $p_2 = n/p(\alpha - \gamma_k)$ to the inner integral in (5.15), we obtain

$$\|B_{N,k}\|_p \leqslant \frac{c}{N^k} \int\limits_{R^n} \frac{\|\Delta_t^{l-k} f\|_{q_k}}{|t|^{n+\alpha}} dt \left\{ \int\limits_{R^n} \frac{dx}{\left(1+\frac{|x|}{N}\right)^{\frac{n}{\alpha-\gamma_k}} \cdots \left(1+\frac{|x-kt|}{N}\right)^{\frac{n}{\alpha-\gamma_k}}} \right\}^{\frac{\alpha-\gamma_k}{n}}. \tag{5.18}$$

We apply (3.10), according to which

$$\|\Delta_t^{l-k} f\|_{q_k} \leqslant c|t|^{\gamma_k} \|\mathbf{D}^{\gamma_k} f\|_{q_k}, \quad f \in I^{\gamma_k}(L_{q_k}),$$

if $l - k > \gamma_k$. Therefore, taking $\gamma_k > l - k$, we obtain

$$\|B_{N,k}f\|_p \leqslant c\|\mathbf{D}^{\gamma_k} f\|_{q_k} \int\limits_{R^n} \frac{dt}{|t|^{n+\alpha-k-\gamma_k}} \left\{ \int\limits_{R^n} \frac{dx}{(1+|x|)^{\frac{n}{\alpha-\gamma_k}} \cdots (1+|x-kt|)^{\frac{n}{\alpha-\gamma_k}}} \right\}^{\frac{\alpha-\gamma_k}{n}}.$$

Applying Lemma 4 to the inner integral (which reduces to the condition $\gamma_k > \alpha - k - 1$), we have

$$\|B_{N,k}f\|_p \leqslant c\|\mathbf{D}^{\gamma_k} f\|_{q_k} \int\limits_{R^n} \frac{|t|^{k+\gamma_k-n-\alpha} dt}{(1+|t|)^{k+1-\alpha+\gamma_k-\varepsilon}} = c_1 \|\mathbf{D}^{\gamma_k} f\|_{q_k} \tag{5.19}$$

under the condition $\alpha - k < \gamma_k$. The uniform bound (5.15) follows from (5.17) and (5.19) under the choice $\gamma_k \in (\alpha - k, l - k) \cap (0, m)$ for $k = 1, 2, \ldots, l-1$ (it is obvious that this intersection is nonempty) and $\gamma_l \in (0, m)$.

2. We verify (5.14) for $f \in C_0^\infty$. Extending (5.18) (which is valid also for $k = l$), we have, by Lemmas 3 and 4,

$$\|B_{N_k} f\|_p \leqslant \frac{c}{N^k} \int_{R^n} \frac{|t|^{l-\alpha-n}\, dt}{(1+|t|)^{l-k}} \left\{ \int_{R^n} \frac{dx}{\left(1+\frac{|x|}{N}\right)^{\frac{n}{\alpha-\gamma_k}} \cdots \left(1+\frac{|x-kt|}{N}\right)^{\frac{n}{\alpha-\gamma_k}}} \right\}^{\frac{\alpha-\gamma_k}{n}}$$

$$\leqslant \frac{c}{N^{k-l+\gamma_k}} \int_{R^n} \frac{dt}{|t|^{n+\alpha-l}(1+|t|)^{k-\alpha+\gamma_k+1-\varepsilon}(1+N|t|)^{l-k}}.$$

If $k = l$, then from the above follows

$$\|B_{N,l} f\|_p \leqslant \frac{c}{N^{\gamma_l}} \int_{R^n} \frac{dt}{|t|^{n+\alpha-l}(1+|t|)^{l-\alpha+\gamma_l+1-\varepsilon}} = \frac{c_1}{N^{\gamma_l}} \to 0.$$

If $k = 1, 2, \ldots, l-1$, then

$$\|B_{N,k} f\|_p \leqslant \frac{c}{N^{k-l+\gamma_k+\mu}} \int_{R^n} \frac{dt}{|t|^{n+\alpha-l+\mu}(1+|t|)^{k-\alpha+\gamma_k+1-\varepsilon}},$$

where $0 \leqslant \mu \leqslant l-k$. The integral on the right converges and the right side approaches zero, if we take $\mu \in (l-k-\gamma_k, \min\{l-k, l-\alpha\})$. Lemma 9 is proved.

REMARK. It follows from (5.12) and (5.15), according to (3.1), that the space $I^\alpha(L_p)$ is invariant under the "reduction" (5.1):

$$\nu_N f \in I^\alpha(L_p), \quad \text{if} \quad f \in I^\alpha(L_p). \tag{5.20}$$

3. *Justification of* (5.3) *for* $\alpha = 1, 3, 5, \ldots$. Let $\alpha = 2m+1$, $m = 1, 2, \ldots$. Since $\nu_N f \in I^\alpha(L_p)$ according to (5.20), the Riesz derivative $\mathrm{D}^{2m+1}(\nu_N f)$ can be replaced by weak derivatives in accordance with (4.14). By boundedness of the Riesz transform in $L_p(R^n)$, $1 < p < \infty$, it remains to show that

$$\left\| \frac{\partial}{\partial x_j} \Delta^m (\nu_N f) \right\|_p \xrightarrow[N\to\infty]{} 0, \quad j = 1, 2, \ldots, n.$$

It is obvious that $\partial \Delta^m(\nu_N f)/\partial x_j$ can be represented in the form

$$\frac{\partial}{\partial x_j} \Delta^m (\nu_N f) = \sum_{|k|+|s|=2m+1} \frac{c_{k,s,j}}{N^{|k|}} (D^k \nu)\left(\frac{x}{N}\right) (D^s f)(x),$$

where the $c_{k,s,j}$ are constants; k and s are multi-indices. Here all the weak derivatives $D^s f$ exist, and $D^s f \in L_{p_{|k|}}$ by Theorem 7. It remains to show that

$$a_{k,s}(N) \overset{\text{def}}{=} \frac{1}{N^{|k|}} \left\| (D^k \nu)\left(\frac{x}{N}\right) (D^s f)(x) \right\|_p \xrightarrow[N\to\infty]{} 0, \quad |k|+|s| = 2m+1. \tag{5.21}$$

For $|k| = 0$ we have

$$a_{0,s}(N) \leqslant \left\{ \int_{|x|>N} |(D^s f)(x)|^p\, dx \right\}^{\frac{1}{p}} \xrightarrow[N\to\infty]{} 0,$$

since $D^s f \in L_p(R^n)$ for $|s| = 2m+1 = \alpha$. Let $|k| > 0$. Applying the Hölder inequality with exponents $p_1 = n/|k|p$ and $p_2 = n/(n-|k|p)$ in (5.21), we obtain

$$a_{k,s}(N) \leqslant \frac{1}{N^{|k|}} \left\{ \int\limits_{N<|x|<2N} \left| (D^k v)\left(\frac{x}{N}\right) \right|^{\frac{n}{|k|}} dx \right\}^{\frac{|k|}{n}}$$

$$\times \left\{ \int\limits_{N<|x|<2N} |(D^s f)(x)|^{p_{|k|}} dx \right\}^{\frac{1}{p_{|k|}}} \leqslant c \left\{ \int\limits_{N<|x|<2N} |(D^s f)(x)|^{p_{|k|}} dx \right\}^{\frac{1}{p_{|k|}}} \xrightarrow[N\to\infty]{} 0,$$

since $D^s f \in L_{p_{|k|}}(R^n)$. Theorem 9 is completely proved.

REMARK. The proof of Theorem 9 on denseness in the excluded case $\alpha = 1, 3, 5, \ldots$, which was carried out by passing to the weak derivative, could have been accomplished entirely within the framework of Riesz derivatives. However, the calculations for this are significantly more complicated. We shall indicate briefly such a proof in the case $\alpha = 1$. The limit passage (5.6) is necessary in the proof. By (1.20) we have

$$B_N f = \frac{1}{N} \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}\left(\frac{x}{N}\right) A_{N,j} f + \Psi_N f,$$

where

$$A_{N,j} f = \int\limits_{R^n} \frac{t_j}{|t|^{n+1}} \frac{N}{|t|+N} f(x-t)\, dt, \qquad \Psi_N f = \int\limits_{R^n} \frac{\psi_N(x,t) f(x-t)}{|t|^{n+1}}\, dt.$$

We omit the bounds for $\Psi_N f$, which are similar to the bounds for $B_N f$ in (5.7)–(5.10) and are based on an application of the Banach-Steinhaus theorem. For a bound on the operator $A_{N,j}$ we use its similarity to the Riesz transform (1.24). We represent it in the form

$$A_{N,j} f = c_n (R_j - \widetilde{P}_{N,j}) f + K_{N,j} f,$$

where $\widetilde{P}_{N,j}$ is the Poisson operator (1.26), and $K_{N,j}$ is the convolution operator $K_{N,j} f = K_{N,j} * f$ with summable kernel

$$K_{N,j}(t) = t_j (|t|^2 + N^2)^{-\frac{n+1}{2}} - t_j |t|^{-n} (|t|+N)^{-1} \in L_1.$$

Applying (1.27), we obtain

$$A_{N,j} f = c_n R_j f - c_n P_N R_j f + K_{N,j} f,$$

and then

$$\left\| \frac{1}{N} \frac{\partial v}{\partial x_j}\left(\frac{x}{N}\right) A_{N,j} f \right\|_p \leqslant c_n \left\{ \int\limits_{N<|x|<2N} |R_j f|^{p_\alpha} dx \right\}^{\frac{1}{p_\alpha}}$$

$$+ \left\{ \int\limits_{R^n} |K_{N,j} f - c_n P_N R_j f|^{p_\alpha}\, dx \right\}^{\frac{1}{p_\alpha}}.$$

Here the first term approaches zero by the fact that $R_j f \in L_{p_\alpha}(R^n)$, and the second by Theorem 5.

§6. A description of $I^{\alpha}(L_p)$ in terms of weights

THEOREM 10. *In order that* $f(x) \in I^{\alpha}(L_p)$, $1 < p < n/\alpha$, *it is necessary and sufficient that*

$$(1+|x|)^{-\alpha} f(x) \in L_p(R^n), \tag{6.1}$$

$$\mathbf{D}^{l\alpha} f \in L_p(R^n), \tag{6.2}$$

where $l > 2[\alpha/2]$ ($l = \alpha$ *for* $\alpha = 1, 3, 5, \ldots$) *in the necessity part and* $l > 2\alpha$ ($\alpha \neq 1, 3, 5, \ldots$) *in the sufficiency part.*

PROOF. *Necessity.* The well-known inequality

$$\int_{R^n} |x|^{-\alpha p} |(K^{\alpha}\varphi)(x)|^p \, dx \leqslant c(n, p, \alpha) \|\varphi\|_p^p, \quad 1 < p < \frac{n}{\alpha}, \tag{6.3}$$

is the multi-dimensional analog of an inequality of Hardy and Littlewood [25] which is established in [43] (compare [31]), and which is a special case of the multi-dimensional generalization of a theorem of Schur, Hardy and Littlewood on integral operators with a homogeneous kernel (see [40], [41], [11] and [21]). It is obvious that (6.1) follows from (6.3), and (6.2) follows from Theorem 2.

Sufficiency. As was shown in the proof of Theorem 4, the representability of $f(x)$ by a Riesz potential turns out to be a consequence of the integral representation (2.13). We shall prove that it holds also under the assumptions (6.1) and (6.2). It is not difficult to see that the proof of Theorem 3 is valid also under the assumptions (6.1) and (6.2) up to formula (2.17). We show that in (2.17) it is possible to pass to the limit in the norm of $L_p(\alpha; R^n)$, if $l > 2\alpha$. In the left side of (2.17) this is obvious in view of the fact that it generates an operator which is bounded from $L_p(R^n)$ into $L_p(\alpha; R^n)$, according to (6.3). The right side represents the operator

$$B_{\varepsilon} f = \frac{1}{\varepsilon^n} \int_{R^n} (\Delta_h^m f)(\xi) \, \mathcal{K}_{l,\alpha}\left(\frac{|x-\xi|}{\varepsilon}\right) d\xi.$$

Here $\Delta_h^m f \in L_p(\alpha; R^n)$, since

$$\|\tau_h f\|_{L_p(\alpha, R^n)} \leqslant (1+|h|)^{\alpha} \|f\|_{L_p(\alpha; R^n)}.$$

The operator B_{ε} is uniformly bounded with respect to ϵ in $L_p(\alpha; R^n)$. In fact, since $l > 2\alpha$, we have $(1+|\xi|)^{\alpha} K_{l,\alpha}(|\xi|) \in L_1(R^n)$ by (2.11), and therefore

$$\|B_{\varepsilon}\|_{L_p(\alpha;R^n) \to L_p(\alpha;R^n)} \leqslant \frac{1}{\varepsilon^n} \int_{R^n} (1+|\xi|)^{\alpha} \left| \mathcal{K}_{l,\alpha}\left(\frac{|\xi|}{\varepsilon}\right) \right| d\xi$$

$$\leqslant \int_{R^n} (1+|\xi|)^{\alpha} |\mathcal{K}_{l,\alpha}(|\xi|)| \, d\xi.$$

By the Banach-Steinhaus theorem it is sufficient to verify passage to the limit in the right side of (2.17) for $f \in C_0^{\infty}$. We have, by (2.8),

$$\left\| \int_{R^n} (\Delta_h^m f)(x-\delta\xi)\, \mathcal{K}_{l,\alpha}(|\xi|)\, d\xi - (\Delta_h^m f)(x) \right\|_{L_p(\alpha;R^n)}$$

$$\leqslant \int_{R^n} |\mathcal{K}_{l,\alpha}(|\xi|)|\, d\xi \left\{ \int_{R^n} (1+|x|)^{-\alpha p} |(\Delta_h^m f)(x-\delta\xi) - (\Delta_h^m f)(x)|^p\, dx \right\}^{\frac{1}{p}} \tag{6.4}$$

$$\leqslant \int_{R^n} \omega_p^1(\Delta_h^m f, \delta\xi)\, |\mathcal{K}_{l,\alpha}(|\xi|)|\, d\xi \to 0$$

by Lebesgue's theorem on passing to a limit, since $\mathcal{K}_{l,\alpha}(|\xi|) \in L_1(R^n)$, and $\Delta_h^m f \in C_0^\infty \subset L_p$. Theorem 10 is proved. (We note that the limit in (6.4) can also be obtained if we use not uniform boundedness of B_ϵ, but the property of continuity "in the whole" of functions from weight spaces, which were established by T. S. Pigolkina [15].)

REMARK. Theorem 10 obviously remains valid if we replace $(1 + |x|)^{-\alpha}$ by $|x|^{-\alpha}$ in its formulation. It is clear that in the necessity part the weight $|x|^{-\alpha}$ is preferable, and in the sufficiency, the weight $(1 + |x|)^{-\alpha}$.

Appendix

We calculate the constant $d_{n,l}(\alpha)$ in (1.6). We have

$$\widehat{T_{l,\epsilon}^\alpha f}(x) = \sum_{k=0}^{l} (-1)^k C_l^k \int_{|t|>\epsilon} \frac{dt}{|t|^{n+\alpha}} \int_{R^n} e^{i(x,\xi)} f(\xi - kt)\, d\xi$$

$$= \hat{f}(x) \sum_{k=0}^{l} (-1)^k C_l^k \int_{|t|>\epsilon} \frac{e^{ik(t,x)}}{|t|^{n+\alpha}}\, dt;$$

that is,

$$\widehat{T_{l,\epsilon}^\alpha f}(x) = \hat{f}(x) \int_{|t|>\epsilon} \frac{[1 - e^{i(t,x)}]^l}{|t|^{n+\alpha}}\, dt.$$

It is not difficult to show that here (at least for $f \in S$) a limit passage in the norm of $L_2(R^n)$ for $l > 2[\alpha/2]$ is possible, so that

$$\widehat{T_l^\alpha f}(x) = \hat{f}(x) \int_{R^n} \frac{[1 - e^{i(t,x)}]^l}{|t|^{n+\alpha}}\, dt.$$

Here for $l \leqslant \alpha < l + 1$ (l is odd) the integral on the right converges conditionally, as can be shown with the help of (1.4). Making the substitution $t = \mathrm{rot}_x(|y|/x)$, we obtain

$$\widehat{T_l^\alpha f}(x) = d_{n,l}(\alpha)\, |x|^\alpha \hat{f}(x),$$

where

$$d_{n,l}(\alpha) = \int_{R^n} \frac{(1 - e^{iy_1})^l}{|y|^{n+\alpha}}\, dy = \int_{-\infty}^{\infty} (1 - e^{iy_1})^l\, dy_1 \int_{R^{n-1}} \frac{dt}{(|t|^2 + y_1^2)^{\frac{n+\alpha}{2}}} =$$

$$= \int_{R^{n-1}} \frac{dt}{(|t|^2+1)^{\frac{n+\alpha}{2}}} \int_{-\infty}^{\infty} \frac{(1-e^{iy})^l}{|y|^{1+\alpha}}\,dy = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)} \int_0^{\infty} \frac{(1-e^{iy})^l + (1-e^{-iy})^l}{y^{1+\alpha}}\,dy$$

$$= \frac{2\pi^{\frac{n-1}{2}}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)} \int_0^{\infty} \frac{\sum_{k=0}^{l} (-1)^k C_l^k \cos ky}{y^{1+\alpha}}\,dy.$$

Let $0 < \alpha < 1$. Using the equality $\Sigma_{k=0}^{l}(-1)^k C_l^k = 0$ and taking account of formula 3.761.4 from [4], we reduce the remaining integral to the form

$$\sum_{k=0}^{l} (-1)^k C_l^k \int_0^{\infty} \frac{\cos ky - 1}{y^{1+\alpha}}\,dy = -\frac{1}{\alpha}\sum_{k=0}^{l} (-1)^k C_l^k \cdot k \int_0^{\infty} \frac{\sin ky}{y^{\alpha}}\,dy$$

$$= \frac{\pi}{2\alpha\Gamma(\alpha)\sin\frac{\alpha\pi}{2}}\,\mathcal{A}_l(\alpha),$$

which yields the value of $d_{n,l}(\alpha)$ indicated in (3). For the remaining values $\alpha \geqslant 1$ the required result can be obtained by analytic continuation with respect to α.

We note that it is easy to show that $d\mathcal{A}_l(\alpha)/d\alpha \neq 0$ in (3) for positive integral values of α.

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