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VARIABLE EXPONENT HARDY AND CARLEMAN-KNOPP INEQUALITIES

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Abstract. The well known $L^p(\mathbb{R}_+) \to L^q(\mathbb{R}_+)$-Hardy inequalities with power weights are known to be extended to the case of variable exponent setting $L^{p(\cdot)}(\mathbb{R}_+) \to L^{q(\cdot)}(\mathbb{R}_+)$. In the case $p(0) = p(\infty)$ and $q(0) = q(\infty)$ we give an estimation of the constants arising in this extension in dependence on the values of $p(\infty)$, $\inf p(x)$ and the values of the exponents A_p, A_q from the decay conditions at the origin and infinity. The obtained estimate enables us to use dilation arguments to derive the variable exponent Carleman-Knopp inequality from the variable exponent Hardy inequality.

რეზიუმე. ჰარდის ცნობილი $L^p(\mathbb{R}_+) \to L^q(\mathbb{R}_+)$ წონიანი უტოლობა ხარისხოვანი წონებით განზოგადებულია ცვლადმაჩვენებლიანი ლებეგის სივრცეებში. როცა $p(0) = p(\infty)$ და $q(0) = q(\infty)$ ჩვენ ვიძლევით შესაბამის უტოლობებში მუდმივების შეფასებას. ეს უკანასკნელი საშუალებას გვაძლევს ცვლადმაჩვენებლიანი ჰარდის უტოლობიდან გამოვიყვანოთ ცვლადმაჩვენებლიანი კარლემან-კნოპის უტოლობა.

1. Introduction

Many classical inequalities have been extended to the case of Lebesgue and other function spaces with variable exponent, for instance, inequalities for Hardy, maximal, singular and potential operators. We refer to the book [2] and surveying articles [11], [12], [16] and recall that the main difficulties in such an extension are caused by the absence of the main classical tools valid for constant exponents. Among them we mention the non-invariance of variable exponent Lebesgue spaces with respect to translations and dilations, the failure of the Young theorem for convolutions, non equivalence of norm and modular inequalities.

The classical Carlemann-Knopp inequality

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \ln|f(t)|\,dt\right)dx \le e\int_0^\infty |f(x)|\,dx \tag{1.1}$$

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was generalized in various directions by many authors, including weighted cases and $p \to q$-versions, see for instance [6], [7], [8], [9], [15] and references therein.

The goal of this paper is to show that in the case of variable exponents there hold the inequalities

$$\left\| \exp\left((1-\alpha)x^{\alpha-1} \int\limits_0^x \frac{\ln f(y)}{y^\alpha}\, dy \right) \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le$$

$$\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}, \ \alpha < 1 - \frac{1}{p(\infty)}, \tag{1.2}$$

$$\left\| \exp\left(\beta x^\beta \int\limits_x^\infty \frac{\ln f(y)}{y^{\beta+1}}\, dy \right) \right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le$$

$$\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}, \ \ 0 < \beta < 1 \tag{1.3}$$

of "Carleman–Knopp type" at the least under some assumptions on $p(0)$ and $p(\infty)$. To this end, we use the known ([7]–[9]) idea of treating the Carleman-Knopp inequality as the limiting case of the Hardy inequality derived by dilation arguments with respect to the exponent p. Note that in the case of variable exponents such limiting cases are even of more interest because of the differences between the modular and norm inequalities.

To exploit this idea, we need Hardy inequalities in variable exponent Lebesgue spaces. Such inequalities have been proved in [3] (see also [1], [5], [13], where the multidimensional versions of the variable exponent Hardy inequalities may be found). These Hardy inequalities were obtained without an estimation of the arising constants . However, to be able to apply Hardy $\to$ Carleman-Knopp dilation arguments, we need to know dependence of these constants on the values of the exponent $p(x)$. By this reason we spend special efforts in this paper to refine the inequalities from [3] by providing an estimation of the constants arising in the variable exponent $L^{p(\cdot)} - L^{q(\cdot)}$-boundedness of the Hardy operators

$$H^{\alpha,\mu} f(x) = x^{\alpha+\mu-1} \int\limits_0^x \frac{f(y)\, dy}{y^\alpha}, \quad \mathcal{H}_{\beta,\mu} f(x) = x^{\beta+\mu} \int\limits_x^\infty \frac{f(y)\, dy}{y^{\beta+1}}\,. \tag{1.4}$$

The paper is organized as follows. In Section 2 we recall some definitions from the theory of variable exponent Lebesgue spaces and refine some known estimates. In Section 3 we give an estimation of the norm of convolution operators $k * f$ in variable exponent Lebesgue spaces, in terms of the constants A_p from the decay condition of the exponent $p(x)$ and the norms $\|f\|_r$ for certain values of r. In Section 4 we apply the results of Section 3 to prove estimates for the constants in variable exponent Hardy inequalities, similarly in terms related to $p(x)$. The estimates of this section with $\mu = 0$

are finally used in the last subsection to prove the result on the inequalities (1.2)–(1.3).

2. Estimation of Norms of Some Embeddings for Variable Exponent Lebesgue Spaces

The notation $\mathcal{P}(\mathbb{R}^n)$ in the sequel stands for the class of variable exponents, i.e. measurable functions $p : \mathbb{R}^n \to [1, \infty]$ (in general, unbounded). The variable exponent space $L^{p(\cdot)}(\mathbb{R}^n)$ with $p \in \mathcal{P}(\mathbb{R}^n)$ is defined in the standard way, see for instance, [2], p. 73:

$$\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 : \int\limits_{\mathbb{R}^n} \left| \frac{f(x)}{\lambda} \right|^{p(x)} \leq 1 \right\}. \tag{2.1}$$

We use the standard notation:

$$p_- = \inf_{x \in \mathbb{R}^n} p(x), \quad p_+ = \sup_{x \in \mathbb{R}^n} p(x).$$

2.1. On known embeddings.

Lemma 2.1. *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ and $q(x) \leq p(x)$ almost everywhere, and*

$$\frac{1}{r(x)} := \frac{1}{q(x)} - \frac{1}{p(x)}\,. \tag{2.2}$$

If $1 \in L^{r(\cdot)}(\mathbb{R}^n)$, then

$$\|f\|_{q(\cdot)} \leq 2^{\frac{1}{q_-}} \, \|1\|_{r(\cdot)} \, \|f\|_{p(\cdot)}. \tag{2.3}$$

Proof. The embedding (2.3) is known (see Lemma 3.3.1 in [2]) in the form

$$\|f\|_{q(\cdot)} \leq 2\|1\|_{r(\cdot)} \, \|f\|_{p(\cdot)}. \tag{2.4}$$

To get (2.3), we use the relation

$$\left\| f^{\frac{1}{\alpha}} \right\|_{\alpha p(\cdot)}^{\alpha} = \|f\|_{p(\cdot)}, \quad \alpha > 0, \tag{2.5}$$

valid for the norm (2.1) with $p \in \mathcal{P}(\mathbb{R}^n)$. Since $\alpha q(x) \leq \alpha p(x)$, we apply the inequality (2.4) to the function $f^{\frac{1}{\alpha}}$ with respect to the norms $\|\cdot\|_{\alpha q(\cdot)}$ and $\|\cdot\|_{\alpha p(\cdot)}$, which is possible when $\alpha \geq \frac{1}{q_-}$ and get $\|f^{\frac{1}{\alpha}}\|_{\alpha q(\cdot)} \leq 2\|1\|_{\alpha r(\cdot)} \|f^{\frac{1}{\alpha}}\|_{\alpha p(\cdot)}$. Then we again use (2.5) and arrive at $\|f\|_{q(\cdot)} \leq 2^{\alpha}\|1\|_{r(\cdot)}\|f\|_{p(\cdot)}$. It remains to choose the best possible value $\alpha = \frac{1}{q_-}$. $\square$

We use the standard norms

$$\|f\|_{X \cap Y} = \max\left\{\|f\|_X, \|f\|_Y\right\}, \quad \|f\|_{X+Y} := \inf_{\substack{f = g+h, \\ g \in X, h \in Y}} \left(\|g\|_X + \|h\|_Y\right)$$

for the intersection $X \cap Y$ and the sum $X + Y := g + h : g \in X, h \in Y$ of two Banach spaces.

Lemma 2.2 (See Theorem 3.3.11 in [2])**.** *Let $p_1, p_2, p_3 \in \mathcal{P}(\mathbb{R}^n)$ and $p_1(x) \le p_2(x) \le p_3(x)$ almost everywhere on $\mathbb{R}^n$. Then*

$$L^{p_1(\cdot)}(\mathbb{R}^n) \cap L^{p_3(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_2(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_1(\cdot)}(\mathbb{R}^n) + L^{p_3(\cdot)}(\mathbb{R}^n) \tag{2.6}$$

with

$$\frac{1}{2}\|f\|_{L^{p_1(\cdot)}+L^{p_3(\cdot)}} \le \|f\|_{L^{p_2(\cdot)}} \le 2^{\frac{1}{(p_1)_-}} \|f\|_{L^{p_1(\cdot)}\cap L^{p_3(\cdot)}}. \tag{2.7}$$

By $\mathcal{P}_\infty(\mathbb{R}^n)$ we denote the class of bounded measurable functions with values in $[1,\infty]$ which satisfy the decay condition for some $p_\infty \in [1,\infty]$. The decay condition will be always taken in the form

$$\left|\frac{1}{p_\infty} - \frac{1}{p(x)}\right| \le \frac{A_p}{\ln(e+|x|)}\,. \tag{2.8}$$

We also write $p_\infty = p(\infty)$ in case $p \in \mathcal{P}_\infty(\mathbb{R}^n)$.

In the sequel, following [2], we use the notation

$$m_\infty(x) = \min\{p(x), p_\infty\} \quad \text{and} \quad M_\infty(x) = \max\{p(x), p_\infty\}.$$

Lemma 2.3. *Let $p \in \mathcal{P}_\infty(\mathbb{R}^n)$ and*

$$\frac{1}{s(x)} := \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|. \tag{2.9}$$

Then

$$L^{M_\infty(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{m_\infty(\cdot)}(\mathbb{R}^n) \tag{2.10}$$

if and only if

$$1 \in L^{s(\cdot)}(\mathbb{R}^n) \tag{2.11}$$

and then

$$\|f\|_{m_\infty(\cdot)} \le 2^{\frac{1}{p_-}} \|1\|_{L^{\widetilde{p}_1(\cdot)}} \|f\|_{p(\cdot)}, \tag{2.12}$$

$$\|f\|_{p(\cdot)} \le 2^{\frac{1}{p_-}} \|1\|_{\widetilde{p}_2(\cdot)} \|f\|_{M_\infty(\cdot)}, \tag{2.13}$$

where $\widetilde{p}_1(x)$ and $\widetilde{p}_2(x)$ are variable exponents defined by

$$\frac{1}{\widetilde{p}_1(x)} := \max\left\{0, \frac{1}{p_\infty} - \frac{1}{p(x)}\right\}, \qquad \frac{1}{\widetilde{p}_2(x)} := \max\left\{0, \frac{1}{p(x)} - \frac{1}{p_\infty}\right\}. \tag{2.14}$$

Proof. This lemma is a slight revision of Lemma 3.3.5 from [2]. The equivalence between (2.10) and (2.11) was proved in Lemma 3.3.5 in [2], the constants arising in (2.12) appear from the arguments there: let $\Pi_+ = \{x \in \mathbb{R}^n : p(x) \ge p_\infty\}$ and $\Pi_- = \mathbb{R}^n \backslash \Pi_+$, so that $\frac{1}{s(x)} = \begin{cases} \frac{1}{\widetilde{p}_1(x)}, & x \in \Pi_+ \\ \frac{1}{\widetilde{p}_2(x)}, & x \in \Pi_- \end{cases}$. As shown in [2] (p. 84), condition (2.11) implies that $1 \in L^{\widetilde{p}_1(\cdot)}(\mathbb{R}^n) \cap L^{\widetilde{p}_2(\cdot)}(\mathbb{R}^n)$, so that Lemma 2.1 is applicable, from which we easily derive (2.12)–(2.13). □

Remark 2.4. Let $p \in \mathcal{P}_\infty(\mathbb{R}^n)$. Then $1 \in L^{s(\cdot)}(\mathbb{R}^n)$ with $s(x)$ defined in (2.9) and embeddings (2.10) with the inequalities (2.12) and (2.13) hold.

Indeed, the decay condition guarantees the validity of embeddings in (2.10), see Section 3.3 in [2]. Consequently, by Lemma 2.3 the decay condition is sufficient for the inclusion (2.11).

2.2. **On constants in the equivalence** $L^{p(\cdot)}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n) \cong L^{p_\infty}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n)$**.** Let $p \in \mathcal{P}_\infty(\mathbb{R}^n)$. The following equivalence

$$L^{p(\cdot)}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n) \cong L^{p_\infty}(\mathbb{R}^n) \cap L^{p_+}(\mathbb{R}^n) \tag{2.15}$$

and the embedding

$$L^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p_\infty}(\mathbb{R}^n) + L^{p_-}(\mathbb{R}^n) \tag{2.16}$$

are known to hold if $1 \in L^{s(\cdot)}(\mathbb{R}^n)$, where $s(x)$ is defined in (2.9), see Lemma 3.3.12 in [2] (see also Lemma 4.5 in [3]). Recall that belongness of p to $\mathcal{P}_\infty$ is sufficient for $1 \in L^{s(\cdot)}(\mathbb{R}^n)$. In the following lemma we specify the constants for the operators of embedding in the statements (2.15) and (2.16).

Lemma 2.5. *Let $p \in \mathcal{P}_\infty(\mathbb{R}^n)$. Then the equivalence* (2.15) *is valid in the form*

$$\|f\|_{L^{p_\infty} \cap L^{p_+}} \le 2^{\frac{2}{p_-}} \|1\|_{\widetilde{p}_1(\cdot)} \|f\|_{L^{p(\cdot)} \cap L^{p_+}}, \tag{2.17}$$

$$\|f\|_{L^{p(\cdot)} \cap L^{p_+}} \le 2^{\frac{1}{p_-}+\frac{1}{p_\infty}} \|1\|_{\widetilde{p}_2(\cdot)} \|f\|_{L^{p_\infty} \cap L^{p_+}} \tag{2.18}$$

and the embedding (2.16) *holds in the form*

$$\|f\|_{L^{p_\infty} + L^{p_-}} \le 2^{1+\frac{1}{p_-}} \|1\|_{\widetilde{p}_1(\cdot)} \|f\|_{p(\cdot)} \tag{2.19}$$

with $2^{1+\frac{1}{p_-}}$ replaced by $2^{\frac{1}{p_-}}$ in the case $p_- = p_\infty$.

Proof. We first observe that the following inequalities are derived from Lemma 2.2:

$$p_\infty \le M_\infty(x) \le p_+ \Longrightarrow \|f\|_{M_\infty(\cdot)} \le 2^{\frac{1}{p_\infty}} \max\left\{\|f\|_{p_+}, \|f\|_{p_\infty}\right\}, \tag{2.20}$$

$$p(x) \le M_\infty(x) \le p_+ \Longrightarrow \|f\|_{M_\infty(\cdot)} \le 2^{\frac{1}{p_-}} \max\left\{\|f\|_{p_+}, \|f\|_{p(\cdot)}\right\}, \tag{2.21}$$

$$m_\infty(x) \le p_\infty \le M_\infty(x) \Longrightarrow \|f\|_{p_\infty} \le 2^{\frac{1}{p_-}} \max\left\{\|f\|_{m_\infty(\cdot)}, \|f\|_{M_\infty(\cdot)}\right\}, \tag{2.22}$$

$$m_\infty(x) \le p(\cdot) \le M_\infty(x) \Longrightarrow \|f\|_{p(\cdot)} \le 2^{\frac{1}{p_-}} \max\left\{\|f\|_{m_\infty(\cdot)}, \|f\|_{M_\infty(\cdot)}\right\}. \tag{2.23}$$

Then the estimate (2.17) is obtained as follows:

$$\begin{aligned}
\max\left\{\|f\|_{p_\infty}, \|f\|_{p_+}\right\} &\le 2^{\frac{1}{p_-}} \max\left\{\|f\|_{m_\infty(\cdot)}, \|f\|_{M_\infty(\cdot)}, \|f\|_{p_+}\right\} \text{ by (2.22)} \\
&\le 2^{\frac{2}{p_-}} \max\left\{\|1\|_{\widetilde{p}_1(\cdot)} \|f\|_{p(\cdot)}, \|f\|_{p(\cdot)}, \|f\|_{p_+}\right\} \text{ by (2.12) and (2.21)} \\
&\le 2^{\frac{2}{p_-}} \|1\|_{\widetilde{p}_1(\cdot)} \|f\|_{L^{p(\cdot)} \cap L^{p_+}}
\end{aligned}$$

with $\|1\|_{\widetilde{p}_1(\cdot)} > 1$ taken into account. Similarly (2.18) is obtained:

$$\max\left\{\|f\|_{p(\cdot)}, \|f\|_{p_+}\right\} \le 2^{\frac{1}{p_-}} \|1\|_{\widetilde{p}_2(\cdot)} \max\left\{\|f\|_{M_\infty(\cdot)}, \|f\|_{p_+}\right\} \qquad \text{by (2.13)}$$

$$\le 2^{\frac{1}{p_-}+\frac{1}{p_\infty}} \|1\|_{L^{\widetilde{p}_2(\cdot)}} \max\left\{\|f\|_{p_\infty}, \|f\|_{p_+}\right\} \qquad \text{by (2.20).}$$

Finally,

$$\|f\|_{L^{p_\infty}+L^{p_-}} \le 2\|f\|_{m_\infty(\cdot)} \qquad \text{by (2.7)}$$

$$\le 2^{1+\frac{1}{p_-}} \|1\|_{L^{\widetilde{p}_1(\cdot)}} \|f\|_{p(\cdot)} \qquad \text{by (2.12),}$$

which proves (2.19).

In the case $p_- = p_\infty$, the factor $2^{1+\frac{1}{p_-}}$ in the last inequality may be replaced by $2^{\frac{1}{p_-}}$, since $m_\infty(x) \equiv p_\infty$ in this case, so that we just have $\|f\|_{L^{p_\infty}+L^{p_-}} = \|f\|_{m_\infty(\cdot)}$. □

2.3. **Estimation of the norms $\|1\|_{L^{r(\cdot)}(\mathbb{R}^n)}$ via the decay constant.** Let the variable exponent $r(x)$ be given by one of the relations

$$\frac{1}{r(x)} = \max\left\{0, \frac{1}{p(x)} - \frac{1}{p_\infty}\right\}, \quad \frac{1}{r(x)} = \max\left\{0, \frac{1}{p_\infty} - \frac{1}{p(x)}\right\}, \quad \frac{1}{r(x)} = \left|\frac{1}{p_\infty} - \frac{1}{p(x)}\right|. \tag{2.24}$$

By $m_0 = m_0(n) \in (n, \infty)$ we denote the unique root of the equation

$$(t-1)(t-2)\cdots(t-n)e^t = |\mathbb{S}^{n-1}|(n-1)!\, e^n. \tag{2.25}$$

Remark 2.6. In the one-dimensional case $n = 1$ one has $m_0 = 1 + \delta$, where $\delta > 0$ is the root of the equation $te^t = 2$, i.e. $m_0 = W(2)$, where W is the Lambert special function. Note that $1,693 \approx 1 + \ln 2 < m_0 < 2$ in this case.

Lemma 2.7. *Let $p \in \mathcal{P}(\mathbb{R}^n)$ satisfy the condition* (2.8) *and $r(x)$ defined by one of the relations in* (2.24). *Then*

$$\|1\|_{L^{r(\cdot)}(\mathbb{R}^n)} \le e^{m_0 A_p}. \tag{2.26}$$

Proof. As observed in the proof of Proposition 4.1.8 in [2], under the condition (2.8) for every $m > 0$ there holds the estimate $\gamma^{r(x)} \le \frac{1}{(e+|x|)^m}$ with $\gamma \le e^{-mA_p}$. Under the choice $m > n$ and $\lambda = e^{mA_p}$ we then have

$$\int\limits_{\mathbb{R}^n} \left(\frac{1}{\lambda}\right)^{r(x)} dx \le \int\limits_{\mathbb{R}^n} \frac{dx}{(e+|x|)^m} =: C_m.$$

Direct calculation gives $C_m = |\mathbb{S}^{n-1}|(n-1)!e^{n-m}\frac{\Gamma(m-n)}{\Gamma(m)} = \frac{|\mathbb{S}^{n-1}|(n-1)!e^{n-m}}{(m-1)(m-2)\cdots(m-n)}$. With the choice $m = m_0$ we have $C_{m_0} = 1$ so that $\int\limits_{\mathbb{R}^n} \left(\frac{1}{\lambda}\right)^{r(x)} \le 1$ with $\lambda = e^{m_0 A_p}$, which proves (2.26). □

3. Estimation of the Norm of Convolution Operators in Variable Exponent Lebesgue Spaces

Let

$$Kf(x) := \int\limits_{\mathbb{R}^n} k(x-y)f(y)\,dy \tag{3.1}$$

be a convolution operator. Theorem 3.1 below is a specification of Theorem 4.6 from [3] with respect to the estimation of the norm of the operator K.

The constant exponents $r_0 \geq 1$ and $s_0 \geq 1$, used in Theorem 3.1 are defined by

$$\frac{1}{r_0} = 1 - \frac{1}{p_\infty} + \frac{1}{q(\infty)}, \qquad \frac{1}{s_0} = 1 - \frac{1}{p_-} + \frac{1}{q_+}, \qquad r_0 \leq s_0. \tag{3.2}$$

We also use the notation

$$\frac{1}{\widetilde{p}_1(x)} := \max\left\{0, \frac{1}{p(\infty)} - \frac{1}{p(x)}\right\}, \quad \frac{1}{\widetilde{q}_2(x)} := \max\left\{0, \frac{1}{q(x)} - \frac{1}{q(\infty)}\right\}. \tag{3.3}$$

Theorem 3.1. *Let $p, q \in \mathcal{P}_\infty(\mathbb{R}^n)$ and $q(\infty) \geq p(\infty)$. If*

$$k \in L^{r_0}(\mathbb{R}^n) \cap L^{s_0}(\mathbb{R}^n), \tag{3.4}$$

then the convolution operator K is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{q(\cdot)}(\mathbb{R}^n) \cap L^{q_+}(\mathbb{R}^n)$ and

$$\|Kf\|_{L^{q(\cdot)} \cap L^{q_+}} \leq \varkappa(k; p, q)\, \|f\|_{L^{p(\cdot)}} \tag{3.5}$$

with

$$\varkappa(k; p, q) = 2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}} \|1\|_{L^{\widetilde{q}_2(\cdot)}} \|1\|_{L^{\widetilde{p}_1(\cdot)}} \max\left\{\|k\|_{L^{r_0}}, \|k\|_{L^{s_0}}\right\} \leq \tag{3.6}$$

$$\leq 2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}} e^{m_0(A_p+A_q)} \max\left\{\|k\|_{L^{r_0}}, \|k\|_{L^{s_0}}\right\}, \tag{3.7}$$

where $m_0 = m_0(n)$ is defined by (2.25) *and $2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}}$ may be replaced by $2^{\frac{2}{p_-}+\frac{1}{p_\infty}}$ in the case $p_- = p_\infty$.*

Proof. Besides (3.2), define

$$\frac{1}{r_1} = 1 - \frac{1}{p_-} + \frac{1}{q(\infty)}, \quad \frac{1}{s_1} = 1 - \frac{1}{p(\infty)} + \frac{1}{q_+}.$$

Then

$$1 \leq r_0 \leq \min\{r_1, s_1\} \leq \max\{r_1, s_1\} \leq s_0 \leq \infty. \tag{3.8}$$

By classical Young's inequality for the convolution operator K, we have

$$\|Kf\|_{q_+} \leq \|k\|_{s_0}\|f\|_{p_-}, \quad \|Kf\|_{q(\infty)} \leq \|k\|_{r_1}\|f\|_{p_-}, \tag{3.9}$$

and

$$\|Kf\|_{q_+} \leq \|k\|_{s_1}\|f\|_{p(\infty)}, \quad \|Kf\|_{q(\infty)} \leq \|k\|_{r_0}\|f\|_{p(\infty)}. \tag{3.10}$$

Therefore,

$$\|Kf\|_{q_+} \leq \|k\|_{L^{s_0}\cap L^{s_1}}\|f\|_{L^{p_-}+L^{p(\infty)}} \tag{3.11}$$

and

$$\|Kf\|_{q(\infty)} \leq \|k\|_{L^{r_0}\cap L^{r_1}} \|f\|_{L^{p_-}+L^{p(\infty)}}. \tag{3.12}$$

Consequently,

$$\|Kf\|_{L^{q_+}\cap L^{q(\infty)}} \leq B\|f\|_{L^{p_-}+L^{p(\infty)}} \tag{3.13}$$

with

$$B := \max\{\|k\|_{L^{s_0}}, \|k\|_{L^{s_1}}, \|k\|_{L^{r_0}}, \|k\|_{L^{r_1}}\} = \max\{\|k\|_{L^{r_0}}, \|k\|_{L^{s_0}}\}, \tag{3.14}$$

where the last equality in (3.14) is a consequence of the continuous embeddings $L^{r_0} \cap L^{s_0} \hookrightarrow L^{r_1} \cap L^{s_0}, L^{r_0} \cap L^{s_0} \hookrightarrow L^{r_0} \cap L^{s_1}$ with the norm of the embedding operator equal to 1. More precisely, $\|k\|_{r_1} \leq \|k\|_{r_0}^t \|k\|_{s_0}^{1-t} \leq \|k\|_{L^{r_0}\cap L^{s_0}}$, where $t = \frac{r_0(s_0-r_1)}{r_1(s_0-r_1)} \in (0,1)$, and then $\|k\|_{L^{r_1}\cap L^{s_0}} \leq \|k\|_{L^{r_0}\cap L^{s_0}}$; similarly, $\|k\|_{L^{r_0}\cap L^{s_1}} \leq \|k\|_{L^{r_0}\cap L^{s_0}}$. Therefore, $\|Kf\|_{L^{q_+}\cap L^{q(\infty)}} \leq 2^{1+\frac{1}{p_-}} B\|1\|_{L^{\tilde{p}_1(\cdot)}}\|f\|_{L^{p(\cdot)}}$ by (2.19). Then by (2.18),

$$\|Kf\|_{L^{q(\cdot)}} \leq 2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}} B\,\|1\|_{L^{\tilde{p}_2(\cdot)}}\,\|1\|_{L^{\tilde{p}_1(\cdot)}}\,\|f\|_{L^{p(\cdot)}}, \tag{3.15}$$

which proves (3.5)-(3.6). The line in (3.7) follows from Lemma 2.7. $\square$

4. Estimation of the Constant in the Variable Exponent Hardy Inequality

4.1. On norms of isomorphism between the spaces $L^{p(\cdot)}(\mathbb{R}_+)$ and $L^{p^*(\cdot)}(\mathbb{R})$. As is well known, the exponential change of variables $x = e^{-\frac{t}{p}} f(e^{-t})$, isometrically maps the space $L^p(\mathbb{R})$ onto $L^p(\mathbb{R}_+)$ and reduces the Hardy operators on $\mathbb{R}_+$ to convolution operators on $\mathbb{R}$ covered by Young theorem (this approach works well in general for integral operators on $\mathbb{R}_+$ with a homogeneous kernel, which goes back to G.Hardy [4], see also details in [10], Subsection 5.1). We will now deal with the mapping

$$(W_p f)(t) = e^{-\frac{t}{p(0)}} f(e^{-t}), \quad t \in \mathbb{R}. \tag{4.1}$$

By $\mathcal{P}_{0,\infty} = \mathcal{P}_{0,\infty}(\mathbb{R}_+)$ we denote the set of all measurable bounded functions $p(x) : \mathbb{R}_+ \to \mathbb{R}_+$ t with $\inf\limits_{x\in\mathbb{R}_+} p(x) \geq 1$, which satisfy the conditions:

i) $0 \leq p_- \leq p(x) \leq p_+ < \infty, \ x \in \mathbb{R}_+$,

ii) there exist p_∞ and p_0 in $(0,\infty)$ such that

$$\sup_{x\in\mathbb{R}_+} \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \ln(e+x) \leq A_p^\infty, \tag{4.2}$$

$$\sup_{0<x\leq\frac{1}{e}} \left|\frac{1}{p(x)} - \frac{1}{p_0}\right| \ln\frac{1}{x} \leq A_p^0 \tag{4.3}$$

(then from (4.17)-(4.3) it also follows that $p_0 \geq 1$ and $p_\infty \geq 1$). It can be easily checked that (4.17)-(4.3) implies that also

$$\sup_{x\in\mathbb{R}_+}\left|\left(\frac{1}{p(x)}-\frac{1}{p_\infty}\right)\ln x\right| \leq A_p, \quad A_p=\max\{A_p^\infty, A_p^0\}. \tag{4.4}$$

We denote

$$p^*(t)=p\left(e^{-t}\right), \quad t\in\mathbb{R}.$$

Note that $p_0=p_\infty \iff p^*(-\infty)=p^*(+\infty)$ and (4.18) is equivalent to

$$\left|\frac{1}{p^*(t)}-\frac{1}{p_\infty}\right| \leq \frac{A_p}{|t|}, \quad t\in\mathbb{R}. \tag{4.5}$$

Note also that from (4.19) it follows that

$$\left|\frac{1}{p^*(t)}-\frac{1}{p_\infty}\right| \leq \frac{A_p}{e\ln(e+|t|)}, \quad t\in\mathbb{R}. \tag{4.6}$$

Lemma 4.1. *Let $p\in\mathcal{P}_{0,\infty}$ and $p_0=p_\infty$. Then the operator W_p maps isomorphically the space $L^{p(\cdot)}(\mathbb{R}_+)$ onto the space $L^{p^*(\cdot)}(\mathbb{R})$ and*

$$e^{-A_p} \leq \|W_p\|_{L^{p^*(\cdot)}(\mathbb{R}_+)\to L^{p(\cdot)}(\mathbb{R})} \leq e^{A_p} \tag{4.7}$$

and

$$e^{-A_p} \leq \|W_p^{-1}\|_{L^{p(\cdot)}(\mathbb{R})\to L^{p^*(\cdot)}(\mathbb{R}_+)} \leq e^{A_p}, \tag{4.8}$$

where $A_p=\max\{A_p^0, A_p^\infty\}$.

Proof. We have

$$\int_{\mathbb{R}}\left|\frac{W_pf(t)}{\lambda}\right|^{p^*(t)} dt = \int_{\mathbb{R}}\left|\frac{e^{-\frac{t}{p(0)}}f\left(e^{-t}\right)}{\lambda}\right|^{p^*(t)} dt =$$

$$= \int_{\mathbb{R}_+}\left|\frac{f(x)}{\lambda x^{\frac{1}{p(x)}-\frac{1}{p(0)}}}\right|^{p(x)} dx. \tag{4.9}$$

From (4.18) it follows that

$$e^{-A} \leq x^{\frac{1}{p(x)}-\frac{1}{p(0)}} \leq e^A. \tag{4.10}$$

Hence

$$\int_{\mathbb{R}_+}\left|\frac{f(x)}{\|W_pf\|_{p^*}e^A}\right|^{p(x)} dx \leq 1 = \int_{\mathbb{R}}\left|\frac{W_pf(t)}{\|W_pf\|_{p^*}}\right|^{p^*(t)} dt \leq$$

$$\leq \int_{\mathbb{R}_+}\left|\frac{f(x)}{\|W_pf\|_{p^*}e^{-A}}\right|^{p(x)} dx, \tag{4.11}$$

which yields (4.21). Inequalities (4.8) are consequences of (4.21).

The possibility to replace $2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}}$ by $2^{\frac{2}{p_-}+\frac{1}{p_\infty}}$ is provided by Lemma 2.5. $\square$

4.2. **Reduction of Hardy inequalities to convolution inequalities.** Let first $\mu \equiv 0$. In the case where α and β are constant, the Hardy operators

$$H^\alpha f(x) = x^{\alpha-1} \int_0^x \frac{f(y)}{y^\alpha} dy \quad \text{and} \quad \mathcal{H}_\beta f(x) = x^\beta \int_x^\infty \frac{\varphi(y)\, dy}{y^{\beta+1}} \tag{4.12}$$

have the kernels, homogeneous of degree -1:

$$k^\alpha(x,y) = \frac{1}{x}\left(\frac{x}{y}\right)^\alpha \theta_+(x-y) \quad \text{and} \quad k_\beta(x,y) = \frac{1}{y}\left(\frac{x}{y}\right)^\beta \theta_+(y-x), \tag{4.13}$$

respectively, where $\theta_+(x) = \frac{1}{2}(1+\operatorname{sign} x)$. It is known that an integral operators $K\varphi(x) = \int_0^\infty k(x,y)\varphi(y)dy$ on $\mathbb{R}^1_+$ with such a kernel may be transformed to a convolution operator on $\mathbb{R}^1$, via the exponential change of variables, see [4]; [10], p. 51, and in the case of constant p, the transformation

$$(W_p f)(t) = e^{-\frac{t}{p}} f(e^{-t}) \ , \quad -\infty < t < \infty \tag{4.14}$$

realizes an isometry of $L^p(\mathbb{R}^1_+)$ onto $L^p(\mathbb{R}^1)$: $\|W_p f\|_{L^p(\mathbb{R}^1)} = \|f\|_{L^p(\mathbb{R}^1_+)}$, and

$$W_p K W_p^{-1} = H, \tag{4.15}$$

where $H\varphi = \int\limits_{\mathbb{R}^1} h(t-\tau)\varphi(\tau)d\tau$, $h(t) = e^{\frac{t}{p'}} k(1,e^t)$, $t \in \mathbb{R}^1$ and $\|h\|_{L^1(\mathbb{R}^n)} = \int\limits_0^\infty y^{-\frac{1}{p}} |k(1,y)| dy$.

In the case of variable exponent $p(x)$ we will use this idea of reducing to convolutions, taking the mapping W_p in the form

$$(W_p f)(t) = e^{-\frac{t}{p(0)}} f(e^{-t}) \ , \quad t \in \mathbb{R}^1. \tag{4.16}$$

To be definite with the constants, we adopt the notation

$$\begin{aligned} A_p^\infty &:= \sup_{x\in\mathbb{R}^1_+} \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \ln(e+x), \\ A_p^0 &:= \sup_{0<x\le\frac{1}{e}} \left|\frac{1}{p(x)} - \frac{1}{p_0}\right| \ln\frac{1}{x}. \end{aligned} \tag{4.17}$$

It can be easily checked that (4.17) implies that also

$$\sup_{x\in\mathbb{R}^1_+} \left|\left(\frac{1}{p(x)} - \frac{1}{p_\infty}\right)\ln x\right| \le \max\{A_p^\infty, A_p^0\}. \tag{4.18}$$

We denote

$$p^*(t) = p\left(e^{-t}\right), \quad t \in \mathbb{R}^1.$$

Note that $p_0 = p_\infty \iff p^*(-\infty) = p^*(+\infty)$ and (4.18) is equivalent to

$$\left|\frac{1}{p^*(t)} - \frac{1}{p_\infty}\right| \leq \frac{1}{|t|} \max\{A_p^\infty, A_p^0\}, \quad t \in \mathbb{R}^1. \tag{4.19}$$

Note also that from (4.19) it follows that

$$\left|\frac{1}{p^*(t)} - \frac{1}{p_\infty}\right| \leq \frac{\max\{A_p^\infty, A_p^0\}}{e\ln(e+|t|)}, \quad t \in \mathbb{R}^1. \tag{4.20}$$

Lemma 4.2. *Let $p \in \mathcal{P}_{0,\infty}$ and $p_0 = p_\infty$. Then the operator W_p maps isomorphically the space $L^{p(\cdot)}(\mathbb{R}^1_+)$ onto the space $L^{p^*(\cdot)}(\mathbb{R}^1)$ and*

$$e^{-A_p} \leq \|W_p\|_{L^{p^*(\cdot)}(\mathbb{R}^1_+)\to L^{p(\cdot)}(\mathbb{R}^1)} \leq e^{A_p}, \tag{4.21}$$

where $A_p = \max\{A_p^0, A_p^\infty\}$.

Proof. We have

$$\int\limits_{\mathbb{R}^1} \left|\frac{W_p f(t)}{\lambda}\right|^{p^*(t)} dt = \int\limits_{\mathbb{R}^1} \left|\frac{e^{-\frac{t}{p(0)}} f\left(e^{-t}\right)}{\lambda}\right|^{p^*(t)} dt =$$

$$= \int\limits_{\mathbb{R}^1_+} \left|\frac{f(x)}{\lambda x^{\frac{1}{p(x)} - \frac{1}{p(0)}}}\right|^{p(x)} dx. \tag{4.22}$$

From (4.18) it follows that $e^{-A} \leq x^{\frac{1}{p(x)} - \frac{1}{p(0)}} \leq e^A$. Hence

$$\int\limits_{\mathbb{R}^1_+} \left|\frac{f(x)}{\|W_p f\|_{p^*} e^A}\right|^{p(x)} dx \leq 1 = \int\limits_{\mathbb{R}^1} \left|\frac{W_p f(t)}{\|W_p f\|_{p^*}}\right|^{p^*(t)} dt \leq$$

$$\leq \int\limits_{\mathbb{R}^1_+} \left|\frac{f(x)}{\|W_p f\|_{p^*} e^{-A}}\right|^{p(x)} dx \tag{4.23}$$

which yields (4.21). □

Lemma 4.3. *For the Hardy operators $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$ with constant α, β and μ the following relations are valid*

$$(W_q H^{\alpha,\mu} W_p^{-1})\psi(t) = \int\limits_{\mathbb{R}^1} h_-(t-\tau)\psi(\tau)d\tau \tag{4.24}$$

and

$$(W_q \mathcal{H}_{\beta,\mu} W_p^{-1})\psi(t) = \int\limits_{\mathbb{R}^1} h_+(t-\tau)\psi(\tau)d\tau, \tag{4.25}$$

where

$$h_-(t) = e^{\left(\frac{1}{p'(0)} - \alpha\right)t}\theta_-(t) \quad \textit{and} \quad h_+(t) = e^{-\left(\frac{1}{p(0)} + \beta\right)t}\theta_+(t), \tag{4.26}$$

q is defined by the condition $\frac{1}{q(0)} = \frac{1}{p(0)} - \mu$ and $\theta_-(t) = 1 - \theta_+(t)$.

Proof. The proof is a matter of direct verification. □

In view of Lemmas 4.3 and 4.2 and Theorem 3.1, we are now able to prove the main result of this Section for Hardy operators. This will be done in the next subsection.

4.3. **Variable exponent Hardy inequalities.** By $\mathcal{M}_{0,\infty}(\mathbb{R}^1_+)$ we denote a class of functions $g \in L^\infty(\mathbb{R}^1_+)$ such that there exist real numbers g_0 and g_∞ such that the following decay conditins

$$|g(x) - g_0| \leq \frac{A}{|\ln x|}, \quad 0 < x \leq \frac{1}{2} \quad \text{and} \quad |g(x) - g_\infty| \leq \frac{A}{\ln x}, \quad x \geq 2$$

hold. We also write $g_0 = g(0)$, $g_\infty = g(\infty)$ in this case. By $\mathcal{P}_{0,\infty}(\mathbb{R}^1_+)$ we denote the subclass of functions in $\mathcal{M}_{0,\infty}(\mathbb{R}^1_+)$ with values in $[1, \infty)$.

The following theorem was in main proved in [3]. We prove it here under more general assumptions.

Theorem 4.4. *Let $\alpha, \beta, \mu \in \mathcal{M}_{0,\infty}(\mathbb{R}^1_+)$, $p \in \mathcal{P}_{0,\infty}(\mathbb{R}^1_+)$ and $p_- > 1$ and*

$$0 \leq \mu(0) < \frac{1}{p(0)} \quad \textit{and} \quad 0 \leq \mu(\infty) < \frac{1}{p(\infty)}.$$

Let also $q(x)$ be any function in $\mathcal{P}_{0,\infty}$ such that

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0) \quad \textit{and} \quad \frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \mu(\infty). \tag{4.27}$$

Then the Hardy-type inequalities

$$\left\| x^{\alpha(x)+\mu(x)-1} \int_0^x \frac{f(y)\,dy}{y^{\alpha(y)}} \right\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \leq C\, \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \tag{4.28}$$

and

$$\left\| x^{\beta(x)+\mu(x)} \int_x^\infty \frac{f(y)\,dy}{y^{\beta(y)+1}} \right\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \leq C \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)}, \tag{4.29}$$

are valid, if and only if α and β satisfy respectively the conditions

$$\alpha(0) < \frac{1}{p'(0)}, \quad \alpha(\infty) < \frac{1}{p'(\infty)}, \tag{4.30}$$

$$\beta(0) > -\frac{1}{p(0)}, \quad \beta(\infty) > -\frac{1}{p(\infty)}. \tag{4.31}$$

Proof. Sufficiency.

1^0. The case where $p(0) = p(\infty)$**,** $\mu(0) = \mu(\infty)$**,** $\alpha(0) - \alpha(\infty)$ **and** $\beta(0) = \beta(\infty)$.

In this case, by the decay condition we have the equivalence

$$x^{\mu(x)} \sim x^{\mu(0)} x^{\alpha(x)} \sim x^{\alpha(0)}, \quad x^{\beta(x)} \sim x^{\beta(0)},$$

on the whole half-axis $\mathbb{R}^1_+$, so that the Hardy operators $H^{\alpha,\mu}$, $\mathcal{H}_{\beta,\mu}$ with variable exponents are equivalent to the Hardy operators with constant exponents $\mu = \mu(0), \alpha = \alpha(0), \beta = \beta(0)$, respectively. To the latter we can apply Lemmas 4.2 and 4.3. We have

$$\|W_p f\|_{L^{p_*(\cdot)}(\mathbb{R}^1)} \sim \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \text{ and } \|W_q^{-1}\psi\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \sim \|\psi\|_{L^{q_*(\cdot)}(\mathbb{R}^1)}, \quad (4.32)$$

where $p_*(t) = p(e^{-t}), q_*(t) = q(e^{-t})$. Therefore, the $L^{p(\cdot)}(\mathbb{R}^1) \to L^{q(\cdot)}(\mathbb{R}^1)$ boundedness of the operators $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$ follows from the $L^{p_*(\cdot)}(\mathbb{R}^1) \to L^{q_*(\cdot)}(\mathbb{R}^1)$ boundedness of the convolution operators on $\mathbb{R}^1$ with the kernels $h_+(t)$ and $h_-(t)$, respectively.

Since $\frac{1}{p'(0)} - \alpha > 0$ and $\frac{1}{p(0)} + \beta > 0$, the convolutions $h_- * \psi$ and $h_+ * \psi$ are bounded operators from $L^{p_*(\cdot)}(\mathbb{R}^1)$ to $L^{q_*(\cdot)}(\mathbb{R}^1)$ in view of Theorem 3.1. Consequently, the Hardy operators $H^{\alpha,\mu}$ and $\mathcal{H}_{\beta,\mu}$ are bounded from $L^{p(\cdot)}(\mathbb{R}^1_+)$ to $L^{q(\cdot)}(\mathbb{R}^1_+)$.

2^0. The general case.

Let $0 < \delta < N < \infty$ and $\chi_E(x)$ denote the characteristic function of a set $E \subset \mathbb{R}^1_+$. We have

$$\begin{aligned} H^{\alpha,\mu} f(x) = &\left(\chi_{[0,\delta]} + \chi_{[\delta,N]} + \chi_{[N,\infty)}\right) H^{\alpha,\mu} \left(\chi_{[0,\delta]} + \chi_{[\delta,N]} + \chi_{[N,\infty)}\right) f(x) = \\ &= \chi_{[0,\delta]}(x) \left(H^{\alpha,\mu}\chi_{[0,\delta]} f\right)(x) + \chi_{[\delta,\infty)}(x) \left(H^{\alpha,\mu}\chi_{[0,N]} f\right)(x) + \\ &+ \chi_{[N,\infty)}(x) \left(H^{\alpha,\mu}\chi_{[N,\infty)} f\right)(x) =: V_1(x) + V_2(x) + V_3(x). \quad (4.33) \end{aligned}$$

It suffices to estimate separately the modulars $I_q(V_k)$, $k = 1, 2, 3$, supposing that $\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \le 1$. For $I_q(V_1)$ we obtain

$$\begin{aligned} I_q(V_1) = &\int_0^\delta \left| \int_0^x \frac{x^{\alpha(x)-1}}{y^{\beta(y)}} f(y) dy \right|^{q(x)} dx \le \\ &\le \int_0^\infty \left(\int_0^x \frac{x^{\alpha_1(x)+\mu_1(x)-1}}{y^{\beta_1(y)}} |f(y)| dy \right)^{q_1(x)} dx = I_{q_1}(H^{\alpha_1,\mu_1} f), \quad (4.34) \end{aligned}$$

where $\alpha_1(x), \mu_1(x)$ and $p_1(x)$ are arbitrarily chosen extensions of the functions $\alpha(x), \mu(x)$ and $p(x)$ from $[0, \delta]$ to the whole half-axis with the preservation of the classes $\mathcal{M}_{0,\infty}(\mathbb{R}^1_+)$ and $\mathcal{P}_{0,\infty}(\mathbb{R}^1_+)$ and such that

$$\alpha_1(\infty) = \alpha(0), \quad \mu_1(\infty) = \mu(0) \quad \text{and} \quad p_1(\infty) = \mu(0).$$

(The possibility of such an extension is in fact obvious, and the direct constriction may be given: $p_1(x) = \omega(x)\, p(x) + (1 - \omega(x))\, p(\infty)$, where

$\omega \in C^\infty([0,\infty))$ has compact support and $\omega(x) = 1$ for $x \in [0,\delta]$ and similarly for $\alpha_1(x)$ and $\mu_1(x)$). From (4.34) we obtain

$$I_q(V_1) \le C < \infty \quad \text{whenever} \quad \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \le 1$$

according to Part $\mathbf{1}^0$ of the proof.

The estimation of $I_q(V_3)$ is quite similar to that of $I_q(V_1)$ with the only difference that the corresponding extension of $p(x)$ must be made from $[N,\infty)$ to $\mathbb{R}^1_+$.

Finally, the estimation of the term $V_2(x)$ is evident:

$$I_q(V_2) \le \int\limits_\delta^\infty \left| x^{\alpha(\infty)-1} \int\limits_0^N \frac{f(y)}{y^{\alpha(0)}} dy \right|^{p(x)} dx, \tag{4.35}$$

where it suffices to apply the Hölder inequality in $L^{p(\cdot)}(\mathbb{R}^1_+)$ when we integrate in y with $\alpha < \frac{1}{p'(0)}$ taken into account, and make use of the fact that $\alpha(\infty) < \frac{1}{p'(\infty)}$ when we integrate in x.

Similarly the case of the operator $\mathcal{H}_\beta$ is considered (or alternatively, one can use the duality arguments, but the latter should be modified by considering separately the spaces on $[0,\delta]$ and $[N,\infty)$, because we admit $p(x) = 1$ in between).

Necessity.

Take $f_0(x) = \dfrac{\chi_{\left[0,\frac{1}{2}\right]}(x)}{x^{\frac{1}{p(0)}} \ln \frac{1}{x}} \in L^{p(\cdot)}(\mathbb{R}^1_+)$ for which the existence of the integral $H^{\alpha,\mu} f_0(x) = x^{\alpha(x)+\mu(x)-1} \int\limits_0^x \dfrac{dy}{y^{\alpha(x)+\frac{1}{p(0)}} \ln \frac{1}{y}} dy$, $0 < x < \frac{1}{2}$ implies the condition $\alpha(0) < \frac{1}{p'(0)}$. For the second choose $f_\infty(x) = \frac{\chi_{[2,\infty)}(x)}{x^\lambda} \in L^{p(\cdot)}(\mathbb{R}^1_+)$, $\lambda > \max(1, 1-\alpha)$. For $x \ge 3$ we have

$$H^{\alpha,\mu} f_\infty(x) \sim x^{\alpha(\infty)+\mu(\infty)-1} \int\limits_2^x \frac{dy}{y^{\alpha(\infty)-\lambda}} \ge$$

$$\ge x^{\alpha(\infty)+\mu(\infty)-1} \int\limits_2^3 \frac{dy}{y^{\alpha+\lambda}} = c x^{\alpha(\infty)+\mu(\infty)-1}$$

which belongs to $L^{p(\cdot)}(\mathbb{R}^1_+)$ only if $\alpha < \frac{1}{p'(\infty)}$.

Similarly the necessity of the conditions (4.31) is proved. $\square$

4.4. **Estimation of the constants in the Hardy inequalities.** Observe, that estimation of constants arising in the boundedness statements in the variable exponent spaces, is not an easy task (it is not always easy even in the case of constant exponents). For variable exponents, they may depend on $p(x)$, for instance, via the constants p_-, p_+ and the constants from the log-condition and decay conditions.

Basing on the calculations made in the prededing sections, we give some estimation of the constants in the Hardy inequalities (4.28)-(4.29) in the cases, where

i) α, β and μ are constants,

ii) $p(0) = p(\infty)$ and $q(0) = q(\infty)$.

Note that in the case where all the exponents p, α, β and μ are constant, the Hardy inequalities (4.28)-(4.29) hold at the least with the constant

$$C = \left(\frac{1-\mu}{\nu}\right)^{1-\mu}, \tag{4.36}$$

where $\nu = \frac{1}{p'} - \alpha$ for the operator $H^{\alpha,\mu}$ and $\nu = \frac{1}{p} + \beta$ for the operator $\mathcal{H}_{\beta,\mu}$ (use the relations (4.24)–(4.25) and apply Young $p \to q$-theorem for convolutions; see [14] for the best constant in (4.28)).

In the following theorem we use the notation $A_p = \max\{A_p^\infty, A_p^0\}$ and similar one for A_q, and denote $\delta = \frac{1}{p_-} - \frac{1}{q_+}$. Recall that the constant $W(2)$ was defined in Remark 2.6, $1 + \ln 2 < W(2) < 2$. Compare formulas (4.5), (4.39) with (4.36).

Theorem 4.5. *Let $p, q \in \mathcal{P}_{0,\infty}, p_0 = p_\infty, q_0 = q_\infty, 0 \leq \mu < \frac{1}{p_\infty}$ and $\frac{1}{q_0} = \frac{1}{p_0} - \mu$. Under the conditions $\alpha < \frac{1}{p'_\infty}, \beta > -\frac{1}{p_\infty}$, the Hardy inequalities (4.28) and (4.29) hold with the constant*

$$C = 2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}} e^{[1+W(2)](A_p+A_q)} \lambda(p,q), \tag{4.37}$$

where

$$\begin{aligned}\lambda(p,q) := \max\left\{\left(\frac{1-\mu}{\frac{1}{p'_\infty} - \alpha}\right)^{1-\mu}, \left(\frac{1-\delta}{\frac{1}{p'_+} - \alpha}\right)^{1-\delta}\right\} \leq \\ \leq \frac{1}{\left(\frac{1}{p'_\infty} - \alpha\right)^{1-\mu}}\end{aligned} \tag{4.38}$$

for the operator $H^{\alpha,\mu}$ and

$$\begin{aligned}\lambda(p,q) := \max\left\{\left(\frac{1-\mu}{\frac{1}{p_\infty} + \beta}\right)^{1-\mu}, \left(\frac{1-\delta}{\frac{1}{p_+} + \beta}\right)^{1-\delta}\right\} \leq \\ \leq \max\left\{1, \frac{1}{\frac{1}{p_\infty} + \beta}\right\}^{1-\mu}\end{aligned} \tag{4.39}$$

for the operator $\mathcal{H}_{\beta.\mu}$; the factor $2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}}$ in (4.37) may be replaced by $2^{\frac{2}{p_-}+\frac{1}{p_\infty}}$ in the case $p_- = p_0 = p_\infty$.

Proof. The estimates will follow from the relations (4.24)–(4.25) and Theorem 3.1 for convolutions. From (4.24) by Lemma 4.2 we have $\|H^{\alpha,\mu}\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leq e^{A_q}\|h_- * W_p f\|_{L^{q^*(\cdot)}(\mathbb{R})}$, where $q^*(t) = q(e^{-t})$. Subsequently, by Theorem 3.1 and Lemma 4.2 again, we obtain

$$\begin{aligned}\|H^{\alpha,\mu}\|_{L^{q(\cdot)}(\mathbb{R}_+)} &\leq e^{A_q}\varkappa(h_-;p^*,q^*)\,\|W_p f\|_{L^{p^*(\cdot)}(\mathbb{R})} \leq \\ &\leq e^{A_p+A_q}\varkappa(h_-;p^*,q^*)\,\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \end{aligned}\tag{4.40}$$

Similarly,

$$\|\mathcal{H}_{\beta,\mu}\|_{L^{q(\cdot)}(\mathbb{R}_+)} \leq e^{A_p+A_q}\varkappa(h_+;p^*,q^*)\,\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \tag{4.41}$$

To estimate the constants $\varkappa(h_\pm;p^*,q^*)$ corresponding to the kernels $h_\pm$, we use (3.6) and obtain

$$\varkappa(h_\pm;p^*,q^*) = 2^{1+\frac{2}{p_-^*}+\frac{1}{p_\infty^*}}\,\|1\|_{L^{\tilde{q}_2^*(\cdot)}}\,\|1\|_{L^{\tilde{p}_1^*(\cdot)}}\,\max\left\{\|h_\pm\|_{L^{r_0^*}},\|h_\pm\|_{L^{s_0^*}}\right\},$$

with

$$\frac{1}{\widetilde{p}_1^*(t)} := \max\left\{0,\frac{1}{p_\infty^*}-\frac{1}{p^*(t)}\right\},\quad \frac{1}{\widetilde{q}_2^*(x)} := \max\left\{0,\frac{1}{q^*(t)}-\frac{1}{q_\infty^*}\right\} \tag{4.42}$$

and

$$\frac{1}{r_0^*} = 1-\frac{1}{p_\infty^*}+\frac{1}{q_\infty^*} = 1-\frac{1}{p_\infty}+\frac{1}{q_\infty} = 1-\mu, \tag{4.43}$$

$$\frac{1}{s_0^*} = 1-\frac{1}{p_-^*}+\frac{1}{q_+^*} = 1-\frac{1}{p_-}+\frac{1}{q_+} = 1-\delta. \tag{4.44}$$

From Lemma 2.7, Remark 2.6 and (4.20) we have $\|1\|_{L^{\tilde{q}_2^*(\cdot)}}\|1\|_{L^{\tilde{p}_1^*(\cdot)}} \leq e^{W(2)(A_p+A_q)}$, so that

$$\varkappa(h_\pm;p^*,q^*) \leq 2^{1+\frac{2}{p_-^*}+\frac{1}{p_\infty^*}}\,e^{W(2)(A_p+A_q)}\,\max\left\{\|h_\pm\|_{L^{r_0^*}},\|h_\pm\|_{L^{s_0^*}}\right\}.$$

Then from (4.40) and (4.41) we arrive at (4.37) with $\lambda(p,q) = \max\left\{\|h_\pm\|_{L^{r_0^*}},\|h_\pm\|_{L^{s_0^*}}\right\}$. It remains to calculate the corresponding norms $\|h_\pm\|$. For constant exponents $\sigma\in[1,\infty)$ we have

$$\|h_\pm\|_{L^\sigma(\mathbb{R})} = \frac{1}{\left(\sigma\gamma_p^\pm\right)^{\frac{1}{\sigma}}} := g_\pm(\sigma),$$

where $\gamma_p^- = \frac{1}{p'_\infty}-\alpha$ and $\gamma_p^+ = \frac{1}{p_\infty}+\beta$ and then

$$\lambda_1(p,q) = \max\{g_-(r_0),g_-(s_0)\},\quad \lambda_2(p,q) = \max\{g_+(r_0),g_+(s_0)\},$$

which gives equalities in (4.5)–(4.39). To justify the inequalities in (4.5)–(4.39), observe that the function $g_\pm(\sigma) = \left(\sigma\gamma_p^\pm\right)^{-\frac{1}{\sigma}}, \sigma\in(0,\infty)$, has minimum at $\sigma_0 = \frac{e}{\gamma_p^\pm}$ equal to $e^{-\frac{\gamma_p^\pm}{e}}$, while $g_\pm(1) = \frac{1}{\gamma_p^\pm}$ and $g_\pm(\infty) = 1$ (Note that the point σ_0 may lie outside the interval $[1,\infty)$ in the case of h_+). Consequently, since $\frac{1}{\gamma_p^-} > 1$, we have $\lambda_1(p,q) \le \frac{1}{\gamma_p^-}$, $\lambda_2(p,q) \le \max\left\{1, \frac{1}{\gamma_p^+}\right\}$. Then finally

$$e^{A_p+A_q}\varkappa(h_-;p^*,q^*) \le \frac{2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}}}{\frac{1}{p'_\infty}-\alpha}\, e^{[1+W(2)](A_p+A_q)},$$

$$e^{A_p+A_q}\varkappa(h_+;p^*,q^*) \le \frac{2^{1+\frac{2}{p_-}+\frac{1}{p_\infty}}}{\min\{1, \frac{1}{p_\infty}+\beta\}}\, e^{[1+W(2)](A_p+A_q)}. \qquad \square$$

Remark 4.6. Note that the exponent $q(x)$ in the above theorem may have values for $0 < x < \infty$ absolutely independent of those of $p(x)$, the only relation between these exponents is imposed at the end points $x = 0$ and $x = \infty$ by the condition $\frac{1}{q_0} = \frac{1}{p_0} - \mu$ and the assumptions $p_0 = p_\infty, q_0 = q_\infty$. In the case we would wish to use a more restrictive "standard" choice $\frac{1}{q(x)} = \frac{1}{p(x)} - \mu$ for all $x \in [1,\infty]$, then $A_q = A_p$, which slightly "simplifies" the above estimation of the norm, but on the other hand, this choice would force us to impose an assumption $\mu < \frac{1}{p_+}$ on μ, more restrictive than the condition (4.27).

4.5. **Knopp-Carleman inequalities in the variable exponent setting.** In this subsection we apply the known dilation procedure to derive the Knopp-Carleman integral inequality with variable exponents from the Hardy inequalities. To apply this procedure on the base of the estimation of the constants in the Hardy inequalities obtained in the previous subsectionm we have to suppose that $p_- = p(0) = p(\infty)$. Recall that we do not suppose that the local log-condition holds.

Theorem 4.7. *Let $p, q \in \mathcal{P}_{0,\infty}$, $p_0 = p_\infty = p_-(:= \inf p(x))$. Then*

$$\left\| \exp\left((1-\alpha)x^{\alpha-1}\int_0^x \frac{\ln f(y)}{y^\alpha}\,dy\right)\right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le$$
$$\le C_p e^{\frac{1}{(1-\alpha)p_\infty}} \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \tag{4.45}$$

for all $\alpha < \frac{1}{p'_\infty}$ and

$$\left\| \exp\left(\beta x^\beta \int_x^\infty \frac{\ln f(y)}{y^{\beta+1}}\,dy\right)\right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le$$
$$\le C_p e^{-\frac{1}{\beta p_\infty}} \|f\|_{L^{p(\cdot)}(\mathbb{R}_+)} \tag{4.46}$$

for all $0 < \beta < B(\delta) - \frac{1}{p_\infty}$, where $B(\delta) = (1-\delta)^{\frac{\delta-1}{\delta}}$, $\delta = \frac{1}{p_-} - \frac{1}{p_+}$ and $C_p = 2^{\frac{3}{p_\infty}} e^{2[1+W(2)]A_p}$.

Note that the $B(\delta)$ appearing in the bound for the exponent β is a decreasing function of $\delta \in (0,1)$, with $B(0)=\infty$ and $B(1)=1$, so that this bound goes to infinity when $p(x)$ becomes constant.

Proof. We rewrite (4.28) with the constant $C=C(p)$ given in (4.37) for the case $\mu=0$ and $p(x)\equiv q(x)$ in the form

$$\left\|(1-\alpha)x^{\alpha-1}\int\limits_0^x \frac{f(y)\,dy}{y^\alpha}\right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le C(p)(1-\alpha)\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}. \tag{4.47}$$

We may assume that $f(x)\ge 0$ and replace $f(x)$ by $f(x)^\lambda$, and also $p(x)$ by $\frac{p(x)}{\lambda}$, where λ is an arbitrary positive number, and make use of the relation

$$\|f^\lambda\|_{p(\cdot)} = \|f\|^\lambda_{\lambda p(\cdot)}. \tag{4.48}$$

We get

$$\left\|\left((1-\alpha)x^{\alpha-1}\int\limits_0^x \frac{f(y)^\lambda\,dy}{y^\alpha}\right)^{\frac{1}{\lambda}}\right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le$$

$$\le \left[(1-\alpha)C_1\left(\frac{p}{\lambda}\right)\right]^{\frac{1}{\lambda}}\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)}. \tag{4.49}$$

Then

$$\left\|\left((1-\alpha)x^{\alpha-1}\int\limits_0^x \frac{f(y)^\lambda\,dy}{y^\alpha}\right)^{\frac{1}{\lambda}}\right\|_{L^{p(\cdot)}(\mathbb{R}_+)} \le$$

$$\le 2^{\frac{3}{p_\infty}} e^{2[1+W(2)]\frac{1}{\lambda}A_{\frac{p}{\lambda}}}\left(\frac{1-\alpha}{1-\alpha-\frac{\lambda}{p_\infty}}\right)^{\frac{1}{\lambda}}\|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \tag{4.50}$$

by Theorem 4.5.

Denote

$$g_\lambda(x) = (1-\alpha)x^{\alpha-1}\int\limits_0^x \frac{f(y)^\lambda\,dy}{y^\alpha}$$

so that $\lim\limits_{\lambda\to 0} g_\lambda(x) = 1$. We have $(g_\lambda(x))^{\frac{1}{\lambda}} = e^{\frac{\ln g_\lambda(x)}{\lambda}}$, and therefore there exists the almost everywhere limit

$$\lim_{\lambda\to 0}(g_\lambda(x))^{\frac{1}{\lambda}} = e^{\lim_{\lambda\to 0}\frac{d}{d\lambda}\ln g_\lambda(x)} = \exp\left((1-\alpha)x^{\alpha-1}\int\limits_0^x \frac{\ln f(y)}{y^\alpha}\,dy\right).$$

By Fatou theorem (see Theorem 2.3.17 in [2] on application of Fatou theorem with respect to variable exponent norm) we may pass to the limit in (4.50) as $\lambda\to 0$. Taking into account that $\frac{1}{\lambda}A_{\frac{p}{\lambda}} = A_p$, we obtain (4.7).

The inequality (4.7) is proved following the same arguments. □

From (4.7) we obtain also the following

Corollary 4.8. *Under the assumptions of Theorem* 4.7 *on* $p(x)$

$$\sup_{0<\beta<1}\left\|\exp\left(\frac{1}{\beta p_\infty}-\beta x^\beta\int\limits_x^\infty\frac{\ln\frac{1}{f(y)}}{y^{\beta+1}}\,dy\right)\right\|_{L^{p(\cdot)}(\mathbb{R}_+)}\leq C_p\|f\|_{L^{p(\cdot)}(\mathbb{R}_+)}.$$

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