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Fractional integrals and hypersingular integrals in variable order Hölder spaces on homogeneous spaces

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Abstracts

We consider non-standard Hölder spaces $H^{\lambda(\cdot)}(X)$ of functions f on a metric measure space (X, d, μ), whose Hölder exponent $\lambda(x)$ is variable, depending on $x \in X$. We establish theorems on mapping properties of potential operators of variable order $\alpha(x)$, from such a variable exponent Hölder space with the exponent $\lambda(x)$ to another one with a "better" exponent $\lambda(x)+\alpha(x)$, and similar mapping properties of hypersingular integrals of variable order $\alpha(x)$ from such a space into the space with the "worse" exponent $\lambda(x) - \alpha(x)$ in the case $\alpha(x) < \lambda(x)$.

These theorems are derived from the Zygmund type estimates of the local continuity modulus of potential and hypersingular operators via such modulus of their densities. These estimates allow us to treat not only the case of the spaces $H^{\lambda(\cdot)}(X)$, but also the generalized Hölder spaces $H^{w(\cdot,\cdot)}(X)$ of functions whose continuity modulus is dominated by a given function $w(x, h), x \in X, h > 0$.

We admit variable complex valued orders $\alpha(x)$, where $\Re\alpha(x)$ may vanish at a set of measure zero.

To cover this case, we consider the action of potential operators to weighted generalized Hölder spaces with the weight $\alpha(x)$.

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1 Introduction

Last decade there was a strong rise of increase of interest to studies of variable spaces, when the parameters defining the space, which are usually constant, may vary from point to point. A typical example is a generalized Lebesgue space with variable exponent defined by the modular $\int\limits_\Omega |f(x)|^{p(x)}\,dx$ (see the surveying papers [2], [11], [24] on this topic), or more generally, Musielak-Orlicz spaces with the Young function also varying from point to point. Another example is the generalized Hölder space of variable order: $\sup\limits_{|h|<t} |f(x+h)-f(x)| \le Ct^{\lambda(x)},\ x\in\mathbb{R}^n$.

Within the frameworks of the Hölder spaces $H^{\lambda(\cdot)}(\Omega)$ with a variable exponent $\lambda(x)$ and more general spaces $H^{w(\cdot,\cdot)}(X)$ with a given variable dominant of continuity modulus of functions, we study mapping properties of potential operators of the form

$$(I^\alpha f)(x) = \int\limits_\Omega \frac{f(y)\,d\mu(y)}{\varrho(x,y)^{N-\alpha(x)}}, \quad x\in\Omega\subset X, \tag{1.1}$$

also of variable order, for functions f defined on an open set of a quasimetric measure space (X,ϱ,μ), where we admit complex values of the variable exponent $\alpha(x)$, $0\le \Re\alpha(x) < 1$, the "dimension" N is the exponent from the growth condition, see (2.4), Ω is an open bounded set in a quasimetric measure space X. We also study the corresponding hypersingular operators

$$(D^\alpha f)(x) = \lim_{\varepsilon\to 0} \int\limits_{y\in\Omega:\varrho(x,y)>\varepsilon} \frac{f(y)-f(x)}{\varrho(x,y)^{N+\alpha(x)}} d\mu(y), \quad x\in\Omega, \tag{1.2}$$

within the frameworks of such spaces. We reveal the mapping properties of the operators I^α and D^α in dependence of local values of $\alpha(x)$ and $\lambda(x)$, including the worsening of the mapping properties when $\Re\alpha(x)$ may tend to zero: we admit that $\Re\alpha(x)$ may be degenerate at some set of points in Ω. We denote

$$\Pi_\alpha = \{x\in\Omega : \Re\alpha(x) = 0\}$$

and suppose that $\mu(\Pi_\alpha) = 0$.

In the case of constant α such kind of problems were widely studied in the case where $X = \mathbb{S}^{n-1}$ for spherical potential operators and related hypersingular integrals and even in a more general setting of generalized Hölder spaces defined by a given (variable) dominant $w(x,h)$ of continuity modulus; we refer to [26], [27], [32], [33], [34] for the case $w = w(h)$, [28], [29], [30] for the case $\omega = h^{\lambda(x)}$, and [31] for the general case $w = w(x,h)$. In the case $X = \mathbb{S}^{n-1}$, the progress was essentially based on the usage of properties of the sphere $\mathbb{S}^{n-1}$, in particular its group properties, which is no more applicable since we do not assume group properties of X.

In the general setting of quasimetric measure spaces (X, ϱ, μ) with growth condition, mapping properties of the operators I^α and D^α in Hölder spaces $H^\lambda(X)$ were studied, in the case of constant λ and constant real α, in [4], [5], [6], [7].

In the variable exponents case, to obtain results stating that the range of the potential operator over a Hölder type space is imbedded into a better space of a similar nature, we use the method of Zygmund type estimates, which also allows to cover the case of the generalized Hölder spaces $H^{\omega(\cdot,\cdot)}(\Omega)$. In the case we study, these estimates are local, depending on points $x \in \Omega$. The same approach is also used for hypersingular integrals. Note that we deal with an open set Ω in X rather than with "the whole" space X, so that the so called cancellation property over Ω, see (3.3), (3.4), in general no more holds. Thus the final statements for potentials depend essentially on the properties of the potential of the constant function. The admission of the case where the cancellation property may fail, is important in application, for instance, to the case of domains Ω in $\mathbb{R}^n$.

It is known that in the case of $X = \mathbb{S}^{n-1}$ and constant α with $0 < \Re\alpha < 1$, the range of the operator (1.1) over a generalized Hölder space with the characteristic $\omega(x,h)$ is isomorphic to a similar space with the "improved" characteristic $h^{\Re\alpha}\omega(x,h)$, this showing a natural improvement of the local smoothness exactly by the order $\Re\alpha$, see [29], [30], [31]. The same is valid for the case $X = \mathbb{R}^n$, if Hölder spaces are considered with power weights $(1+|x|)^\gamma$ at infinity, see [26]. In the general setting of quasimetric measure spaces, we may obtain statements on the mapping properties of the type

$$I^\alpha : H^{\lambda(\cdot)}(\Omega) \to H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega),$$

and separately $D^\alpha : H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega) \to H^{\lambda(\cdot)}(\Omega)$. However, these two statements in general do not provide the isomorphisn $I^\alpha(H^{\lambda(\cdot)}(\Omega)) = H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$, since D^α and I^α are not inverse operators in general. Recall that when $\alpha = const$ and $\Omega = X = \mathbb{R}^n$ or $\Omega = X = \mathbb{S}^{n-1}$, we have $D^\alpha I^\alpha = cI$ with some constant factor c, see [23], which no more holds when α is

variable or X is a more general set. As shown in [5] for constant α, the composition $D^\alpha I^\alpha$, in case of metric measure space X with cancellation property, is a Calderon-Zygmund operator with standard kernel. We also refer to [22], where in the one-dimensional case $X = \mathbb{R}^1$, but for variable $\alpha(x)$, it was shown that the composition $D_+^\alpha I_+^\alpha$ of Liouville fractional operators is an invertible operator of the form $I + T$ with compact T.

The paper is organized as follows. Section 2 contains necessary preliminaries, including estimations of some integrals of the form $\int\limits_\Omega f[\varrho(x,y)]d\mu(y)$ in terms of one-dimensional integrals, which replaces in a sense the passage to polar coordinates typical for the case $X = \mathbb{R}^n$. It also includes definition of generalized Hölder spaces with variable characteristics on a quasimetric measure space, as well as definition of variable Bary-Stechkin classes of characteristics for these spaces. Section 3 contains the main results.

By C, c we denote various absolute constants which do not depend on $x \in X$. Note that we pay an attention to estimation of arising constants, more careful than usual, because of variable exponents and the possibility for $\Re\alpha(x)$ to degenerate at some set.

2 Preliminaries

2.1 Notation and two technical lemmas

Let (X, ϱ, μ) be a quasimetric measure space with measure μ and the quasidistance ϱ, i.e. a function $\varrho : X \times X \to [0, \infty)$ which satisfies the conditions

$$\varrho(z,y) = 0 \Longleftrightarrow z = y, \quad \varrho(x,y) = \varrho(y,x), \quad \text{for all} \quad x \text{ and } y \text{ in } X, \tag{2.1}$$

$$\varrho(x,y) \le k[\varrho(x,z) + \varrho(z,y)], \quad k \ge 1. \tag{2.2}$$

The space (X, ϱ, μ) is called homogeneous if the measure satisfies the doubling condition $\mu B(x, 2r) \le C\mu B(x, r)$. We refer, for instance to [3] for basics on homogeneous spaces.

As was shown in [12], a homogeneous space (X, ϱ, μ) admits an equivalent quasimetric ϱ_1 for which there exists an exponent $\theta \in (0, 1]$ such that the property

$$|\varrho_1(x,z) - \varrho_1(y,z)| \le M\varrho_1^\theta(x,y)\left\{\varrho_1(x,z) + \varrho_1(y,z)\right\}^{1-\theta} \tag{2.3}$$

holds. When ϱ is a metric, then ϱ automatically satisfies property (2.3) with $\theta = 1$ and $M = 1$.

Definition 2.1. We say that the quasimetric ϱ is regular of order $\theta \in (0, 1]$, if it itself satisfies property (2.3). (This notion does not preassume that (X, ϱ, μ) is homogeneous).

In the sequel we suppose that all the balls $B(x,r) = \{y \in X : \varrho(x,y) < r\}$ are measurable and $\mu S(x,r) = 0$ for all the spheres $S(x,r) = \{y \in X : \varrho(x,y) = r\}, x \in X,\ r \geq 0$. We also suppose that measures of balls satisfy the condition

$$\mu B(x,r) \leq Kr^N \quad \text{as} \quad r \to 0, \quad K > 0, \tag{2.4}$$

where $N > 0$ need not be an integer.

Let Ω be an open set in X and $d = diam\,\Omega$. By $WL(\Omega)$ we denote the class of functions f defined on Ω satisfying the weak Lipshitz condition

$$|f(x) - f(y)| \leq \frac{A}{\ln \frac{1}{\varrho(x,y)}}, \quad \varrho(x,y) \leq \frac{\min(1,d)}{2},\ x,y \in \Omega, \tag{2.5}$$

where the constant $A = A(f) > 0$ does not depend on x and y.

We say that a non-negative function $L(x,t)$ defined on $\Omega \times [0,d], 0 < d \leq \infty$, is almost increasing in t uniformly in x, if there exists a constant $C_L \geq 1$ such that

$$L(x,t) \leq C_L L(x,\tau) \quad \text{for all} \quad 0 < t < \tau < d.$$

Everywhere below we take $C_L = \sup\limits_{x\in\Omega} \sup\limits_{0<t<\tau<d} \frac{L(x,t)}{L(x,\tau)}$.

Let $a > 1$ and

$$\mathfrak{m}_a(t) = \begin{cases} \frac{a^t-1}{t}, & t \neq 0 \\ \ln a, & t = 0 \end{cases}.$$

It is easy to check that $\mathfrak{m}(t)$ is increasing for all $t \in \mathbb{R}^1$.

Lemma 2.2. *Let $L(x,t)$ be a non-negative function defined on $\Omega \times [0,d], 0 < d \leq \infty$, almost increasing in t uniformly in x, and $\gamma(x)$ an arbitrary real-valued function. Then*

$$\sum_{k=0}^{\infty} \frac{L\left(x, a^{-k}r\right)}{\left(a^{-k}r\right)^{\gamma(x)}} \geq \frac{1}{C_L \mathfrak{m}_a[\gamma(x)]} \int_0^r \frac{L(x,t)}{t^{\gamma(x)}} \frac{dt}{t} \tag{2.6}$$

and

$$\sum_{k=0}^{\infty} \frac{L\left(x, a^{-k}r\right)}{\left(a^{-k}r\right)^{\gamma(x)}} \leq \frac{C_L a^{\gamma(x)}}{\mathfrak{m}_a[\gamma(x)]} \int_0^r \frac{L(x,t)}{t^{\gamma(x)}} \frac{dt}{t} + \frac{L(x,r)}{r^{\gamma(x)}}, \tag{2.7}$$

where $a > 1$, $0 < r < d$ and $x \in \Omega$. If $L(x,t)$ satisfies the "doubling-type" condition

$$L(x,at) \leq D_L(a) L(x,t), \quad a > 1 \tag{2.8}$$

where $D_L(a) > 0$ does not depend on t (but may in general depend on x), then (2.7) is valid also in the form

$$\sum_{k=0}^{\infty} \frac{L\left(x, a^{-k}r\right)}{\left(a^{-k}r\right)^{\gamma(x)}} \leq \frac{C_L D_L(a)}{\mathfrak{m}_a[\gamma(x)]} \int_0^r \frac{L(x,t)}{t^{\gamma(x)}} \frac{dt}{t}. \tag{2.9}$$

Proof. We have

$$\int\limits_{a^{-k-1}r}^{a^{-k}r} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t} \le C_L L(x,a^{-k}r)G_k(x),$$

where $G_k(x) = \int\limits_{a^{-k-1}r}^{a^{-k}r} t^{-\gamma(x)-1}dt$. Treating separately the cases where $\gamma(x) \neq 0$ and $\gamma(x) = 0$, we see that

$$G_k(x) = \frac{\mathfrak{m}_a[\gamma(x)]}{(a^{-k}r)^{\gamma(x)}} \tag{2.10}$$

in both the cases. Therefore,

$$\int\limits_{a^{-k-1}r}^{a^{-k}r} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t} \le C_L \mathfrak{m}_a[\gamma(x)]\frac{L(x,a^{-k}r)}{(a^{-k}r)^{\gamma(x)}}$$

and we arrive at (2.6). To prove (2.7), we again use the almost monotonicity of $L(x,t)$ and have

$$\int\limits_{a^{-k-1}r}^{a^{-k}r} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t} \ge \frac{L(x,a^{-k-1}r)}{C_L}G_k(x) = \frac{\mathfrak{m}_a[\gamma(x)]}{C_L}\frac{L(x,a^{-k-1}r)}{(a^{-k}r)^{\gamma(x)}}.$$

Therefore,

$$\sum_{k=0}^{\infty} \frac{L(x,a^{-k-1}r)}{(a^{-k}r)^{\gamma(x)}} \le \frac{C_L}{\mathfrak{m}_a[\gamma(x)]}\int\limits_0^r \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t}. \tag{2.11}$$

Since

$$\sum_{k=0}^{\infty} \frac{L(x,a^{-k}r)}{(a^{-k}r)^{\gamma(x)}} = a^{\gamma(x)}\sum_{k=0}^{\infty} \frac{L(x,a^{-k-1}r)}{(a^{-k}r)^{\gamma(x)}} + \frac{L(x,r)}{r^{\gamma(x)}},$$

we arrive at (2.7). Inequality (2.9) follows immediately from (2.11) by (2.8). □

Remark 2.3. The possibility to choose an arbitrary $a > 1$ in lemma 2.2 will be later used in applications of this lemma in order to optimize constants in some inequalities.

Lemma 2.4. *Let $L(x,r)$ and $\gamma(x)$ be as in Lemma 2.2 and $a > 1$. The inequalities are valid*

$$\frac{1}{C_L\mathfrak{m}_a[\gamma(x)]}\int\limits_r^{\frac{d}{a}} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t} \le \sum_{k=1}^{\left[\log_a \frac{d}{r}\right]} \frac{L\left(x,a^k r\right)}{(a^k r)^{\gamma(x)}} \le \frac{C_L D_L(a)}{\mathfrak{m}_a[\gamma(x)]}\int\limits_r^d \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t}, \quad x \in \Omega, \tag{2.12}$$

where it is also assumed that $L(x,t)$ satisfies the doubling condition (2.8) in the case of the right hand side inequality, and $0 < r \le \frac{d}{a}$ in the left-hand side inequality and $0 < r < d$ in the right-hand side one.

Proof. Since $L(x,t)$ is almost increasing in t, we have

$$\int\limits_{a^{k-1}r}^{a^k r} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t} \le C_L L(x,a^k r) \int\limits_{a^{k-1}r}^{a^k r} t^{-\gamma(x)-1}dt = C_L \mathfrak{m}_a[\gamma(x)]\frac{L(x,a^k r)}{(a^k r)^{\gamma(x)}}$$

by (2.10). Hence

$$\sum_{k=1}^{\left[\log_a \frac{d}{r}\right]} \frac{L(x,a^k r)}{(a^k r)^{\gamma(x)}} \ge \frac{1}{C_L \mathfrak{m}_a[\gamma(x)]} \sum_{k=1}^{\left[\log_a \frac{d}{r}\right]} \int\limits_{a^{k-1}r}^{a^k r} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t} = \frac{1}{C_L \mathfrak{m}_a[\gamma(x)]} \int\limits_{r}^{da^{-\eta}} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t}$$

where

$$\eta = \eta(r) = \log_a \frac{d}{r} - \left[\log_a \frac{d}{r}\right] \in [0,1). \tag{2.13}$$

Since $da^{-\eta} \ge \frac{d}{a}$, we arrive at the left-hand side inequality in (2.12). To obtain the inverse inequality, we again use the almost monotonicity of $L(x,t)$ and have

$$\int\limits_{a^{k-1}r}^{a^k r} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t} \ge \frac{1}{C_L} L(x,a^{k-1}r) \int\limits_{a^{k-1}r}^{a^k r} t^{-\gamma(x)-1}dt = \frac{\mathfrak{m}_a[\gamma(x)]}{C_L}\frac{L(x,a^{k-1}r)}{(a^k r)^{\gamma(x)}}.$$

Therefore,

$$\sum_{k=1}^{\left[\log_a \frac{d}{r}\right]} \frac{L(x,a^{k-1}r)}{(a^k r)^{\gamma(x)}} \le \frac{C_L}{\mathfrak{m}_a[\gamma(x)]} \sum_{k=1}^{\left[\log_a \frac{d}{r}\right]} \int\limits_{a^{k-1}r}^{a^k r} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t} \le \frac{C_L}{\mathfrak{m}_a[\gamma(x)]} \int\limits_{r}^{d} \frac{L(x,t)}{t^{\gamma(x)}}\frac{dt}{t}$$

and we arrive at the right-hand side inequality. □

2.2 Estimation of truncated potential type integrals via one-dimensional integrals

Lemmas 2.5 and 2.8 given below provide in sense a replacement of the formula of the passage to polar coordinates used in the case $X = \mathbb{R}^n$.

Lemma 2.5. *Let $L(x,t)$ be a non-negative function defined on $\Omega \times [0,d], 0 < d \le \infty$, almost increasing in t uniformly in x. If X satisfies condition (2.4) and $\nu(x)$ is an arbitrary real-valued non-negative function, then*

$$\int\limits_{B(x,r)} \frac{L[x,\varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) \le C_a(x) a^{\nu(x)-N} \int\limits_0^r \frac{t^{N-1}L(x,t)}{t^{\nu(x)}}dt + C_L K a^{\nu(x)} \frac{L(x,r)}{r^{\nu(x)-N}}, \tag{2.14}$$

where $x \in \Omega,\ 0 < r < d$,

$$C_a(x) = \frac{KC_L^2 a^{\nu(x)}}{\mathfrak{m}_a(\nu(x) - N)}, \tag{2.15}$$

and $a > 1$ *is an arbitrarily chosen number. If* $L(x,t)$ *additionally satisfies condition* (2.8)*, then* (2.14) *is valid also in the form*

$$\int\limits_{B(x,r)} \frac{L[x, \varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) \le C(x) \int\limits_0^r \frac{t^{N-1} L(x,t)}{t^{\nu(x)}} dt, \quad x \in \Omega,\ 0 < r < d, \tag{2.16}$$

where $C(x) = \min_{a>1} C_a(x) D_L(a)$*. In the cases where* $D_L(a)$ *has a power growth, i.e.* $D_L(a) = D_L a^{\beta(x)}, D_L = const, \beta(x) \ge 0$*, we have*

$$C(x) = KC_L^2 D_L[N + \beta(x)] \left(\frac{\nu(x) + \beta(x)}{N + \beta(x)} \right)^{\frac{\nu(x)+\beta(x)}{\nu(x)-N}}, \tag{2.17}$$

with $\left(\frac{\nu(x)+\beta(x)}{N+\beta(x)} \right)^{\frac{\nu(x)+\beta(x)}{\nu(x)-N}} \Bigg|_{\nu(x)=N} = e.$

Proof. We have

$$\int\limits_{B(x,r)} \frac{L[x, \varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) = \sum_{k=0}^{\infty} \int\limits_{a^{-k-1}r < \varrho(x,y) < a^{-k}r} \frac{L\left[x, \varrho(x,y)\right]}{\varrho(x,y)^{\nu(x)}} d\mu(y)$$

$$\le C_L K \sum_{k=0}^{\infty} \frac{L\left(x, a^{-k}r\right)}{(a^{-k-1}r)^{\nu(x)}} \left(a^{-k}r\right)^N = C_L K a^{\nu(x)} \sum_{k=0}^{\infty} \frac{L\left(x, a^{-k}r\right)}{(a^{-k}r)^{\nu(x)-N}}.$$

Then by (2.7) and (2.9) we arrive at inequalities (2.14)-(2.16). In the cases where $D_L(a) = D_L a^{\beta(x)}$, we can minimize the constant $C_a(x) D_L(a)$. Direct calculation shows that the minimum is attained at $a = \left(\frac{\nu(x)+\beta(x)}{N+\beta(x)} \right)^{\frac{1}{\nu(x)-N}}$ for those x where $\nu(x) \ne N$ and $a = e^{\frac{1}{N+\beta(x)}}$ when $\nu(x) = N$. After easy calculations this gives the "constant" (2.17).

□

Remark 2.6. Lemma 2.5 will be applied in the sequel to the case where $L(x,t) = \omega(f,x,t)$ is the local continuity modulus. As is well known, condition (2.8) in this case holds with $D_L(a) = [a] + 1 \le a + 1 < 2a$.

Corollary 2.7. *Let condition* (2.4) *be satisfied and* $\alpha(x) \ge 0, x \in \Omega$*. Then for all the points* x*, where* $\alpha(x) > 0$*, the estimate holds*

$$\int\limits_{\varrho(x,z)<r} \frac{d\mu(z)}{\varrho(x,z)^{N-\alpha(x)}} \le KC_{\alpha,N}(x) \frac{r^{\alpha(x)}}{\alpha(x)} \tag{2.18}$$

where $C_{\alpha,N}(x) = N\left[1 - \frac{\alpha(x)}{N}\right]^{1-\frac{N}{\alpha(x)}}$ *and* $N \le c_{\alpha,N}(x) \le e[N + \alpha(x)]$.

Proof. Estimate (2.18) is derived from (2.16)-(2.17) with $L(x,t) \equiv 1$ and $\nu(x) = N - \alpha(x)$. The left-hand side bound $N \le c_{\alpha,N}(x)$ is obvious, while the right-hand side one may be obtained from the inequality $\frac{1-t}{1+t} \le e(1-t)^{\frac{1}{t}}, t \in [0,1]$, which may be verified by standard tools of analysis.

Observe that estimates of the type (2.18) are known in the case of constant $\alpha(x) = const$ with some constant in the inequality (see for instance, [5], Lemma 1); our goal was to obtain the constant explicitly dependent on the parameters involved, including dependence on the values of $\alpha(x)$ which may tend to zero. Note that in the Euclidean case $X = \mathbb{R}^N$ inequality (2.18) holds with $K = \frac{|S^{N-1}|}{N}$ and $C_{\alpha,N} = N$. □

Lemma 2.8. *Let X satisfy condition* (2.4)*, $L(x,t)$ be as in Lemma 2.2 and fulfill the doubling condition* (2.8) *and let $\nu(x)$ be as in Lemma 2.5. Then*

$$\int\limits_{\Omega\setminus B(x,r)} \frac{L[x,\varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) \le \frac{KC_L^2 D_L(a) a^{\nu(x)}}{\mathfrak{m}_a[\nu(x) - N]} \int\limits_r^d \frac{t^{N-1} L(x,t)}{t^{\nu(x)}} dt + KC_L a^{\nu(x)} \frac{L(x,d)}{d^{\nu(x)-N}}, \quad (2.19)$$

where $0 < r < d$, the second term on the right-hand side being absent in the case $d = \infty$. When $d < \infty$, estimate (2.19) *may be also given in the form*

$$\int\limits_{\Omega\setminus B(x,r)} \frac{L[x,\varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) \le 2C_a(x) D_L(a) \int\limits_r^d \frac{t^{N-1} L(x,t)}{t^{\nu(x)}} dt, \quad 0 < r < \frac{d}{a}, \quad (2.20)$$

where $C_a(x)$ is the same as in (2.15)*. In the cases where $D_L(a)$ has a power growth, i.e. $D_L(a) = D_L a^{\beta(x)}, D_L = const, \beta(x) \ge 0$, and $\nu(x) \ge N$, estimate* (2.20) *may be optimized as follows*

$$\int\limits_{\Omega\setminus B(x,r)} \frac{L[x,\varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) \le 2C(x) \int\limits_r^d \frac{t^{N-1} L(x,t)}{t^{\nu(x)}} dt, \quad 0 < r < e^{-\frac{1}{N}} d, \quad (2.21)$$

where $C(x)$ is the same as in (2.16).

Proof. Note that estimate (2.20) with $a = 2$ was proved in [9] for functions $L(x,t)$ of the form $L(x,t) = \left[\frac{g(t)}{t^N}\right]^{p(x)}$ with an almost increasing $g(t)$, without explicit evaluation of the factor $C_a(x)$. We have

$$\int\limits_{X\setminus B(x,r)} \frac{L[x,\varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) \le \int\limits_{r<\varrho(x,z)<d} \frac{L[x,\varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) =$$

$$\sum_{k=1}^{\left[\log_a \frac{d}{r}\right]} \int\limits_{a^{k-1}r<\varrho(x,z)<a^k r} \frac{L[x,\varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) \quad + \int\limits_{a^{-\eta}d<\varrho(x,z)<d} \frac{L[x,\varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z)$$
$$=: F_1(x,r) + F_2(x,r),$$

where $F_2(x,r) \equiv 0$ in the case $d = \infty$ and $\eta \in [0,1)$ is the same as in (2.13). For $F_1(x,r)$ by the almost monotonicity of $L(x,r)$ we obtain

$$F_1(x,r) \le C_L K \sum_{k=1}^{\left[\log_a \frac{d}{r}\right]} \frac{L(x,a^k r)}{(a^{k-1}r)^{\nu(x)}} \left(a^k r\right)^N = C_L K a^{\nu(x)} \sum_{k=1}^{\left[\log_a \frac{d}{r}\right]} \frac{L\left(x,a^k r\right)}{(a^k r)^{\nu(x)-N}}.$$

Then

$$F_1(x,r) \le \frac{K C_L^2 D_L(a) z^{\nu(X)}}{\mathfrak{m}_a[\nu(x)-N]} \int_r^d \frac{t^{N-1} L(x,t)}{t^{\nu(x)}} dt$$

by (2.12) with $\gamma(x) = \nu(x) - N$. For $F_2(x,r)$ we observe that $\frac{d}{a} \le a^{-\eta} d$, so that

$$F_2(x,r) \le \int\limits_{\frac{d}{a}<\varrho(x,z)<d} \frac{L[x,\varrho(x,z)]}{\varrho(x,z)^{\nu(x)}} d\mu(z) \le K C_L a^{\nu(x)} \frac{L(x,d)}{d^{\nu(x)-N}}$$

and we arrive at (2.19). To obtain (2.20) from (2.19), we observe that for $r < \frac{d}{a}$

$$\int_r^d \frac{L(x,t)}{t^{\nu(x)-N+1}} dt \ge \int_{\frac{d}{a}}^d \frac{L(x,t)}{t^{\nu(x)-N+1}} dt \ge \frac{1}{C_L} L\left(x, \frac{d}{a}\right) \int_{\frac{d}{a}}^d \frac{1}{t^{\nu(x)-N+1}} dt$$

$$= \frac{1}{C_L} L\left(x, \frac{d}{a}\right) \frac{\mathfrak{m}[(\nu(x)-N)_+]}{d^{\nu(x)-N}} \ge \frac{L(x,d)}{C_L D_L(a)} \frac{\mathfrak{m}[\nu(x)-N]}{d^{\nu(x)-N}}$$

by formula (2.10) with $k = 0, r = d$ and $\gamma(x) = \nu(x) - N$, and assumption (2.8), which yields (2.20).

Finally, to arrive at (2.21), we minimize $C_a(x) D_L(a)$ as in the end of the proof of Lemma 2.5 and observe that for the minimizing value $a = \left(\frac{\nu(x)+\beta(x)}{N+\beta(x)}\right)^{\frac{1}{\nu(x)-N}}$ one has $\frac{d}{a} \ge e^{-\frac{1}{N}} d$. □

Lemma 2.9. *Let $x, y, z \in X$, $\varrho(x,z) \ge 2\varrho(x,y)$ and $\Re\gamma \ge -1$.*

I) If $\varrho(x,y)$ is a metric, then

$$\left|\varrho(x,z)^{-\gamma} - \varrho(y,z)^{-\gamma}\right| \le 2^{\Re\gamma+1} |\gamma| \frac{\varrho(x,y)}{\varrho(x,z)^{\Re\gamma+1}}. \tag{2.22}$$

II) If $\varrho(x,y)$ is a regular quasimetric of order $\theta \in (0,1]$, then

$$\left|\varrho(x,z)^{-\gamma} - \varrho(y,z)^{-\gamma}\right| \le C_\gamma \frac{\varrho^\theta(x,y)}{\varrho(x,z)^{\Re\gamma+\theta}}, \tag{2.23}$$

where $C_\gamma = M|\gamma|2^{\Re\gamma+1}3^{1-\theta}$ and M is the constant from (2.3).

Proof. Inequalities of the lemma are in fact well known, see for instance [7], but we dwell on some details of the proof since we admit complex-valued exponents γ and are interested in evaluation of the arising constant C_γ. Inequality (2.22) is an immediate consequence of the numerical inequality

$$|b^{-\gamma} - c^{-\gamma}| \le |\gamma| \cdot |b-c|(\min\{b,c\})^{-\Re\gamma-1}, \quad b>0,\ c>0, \quad \gamma \in \mathbf{C}. \tag{2.24}$$

(see its proof in Appendix). In the case $b \ge 2|a-b|$, from (2.24) we easily obtain

$$|b^{-\gamma} - c^{-\gamma}| \le 2^{\Re\gamma+1}|\gamma| \cdot |b-c|b^{-\Re\gamma-1}. \tag{2.25}$$

Hence with $a = \varrho(x,z)$ and $b = \varrho(y,z)$, inequality (2.22) follows when ϱ is a metric. In the case where ϱ is a regular quasimetric of order $\theta \in (0,1]$, inequality (2.23) follows from (2.25) in view of (2.3). □

2.3 Hölder and generalized Hölder spaces with variable characteristics on a quasimetric measure space

For fixed $x \in \Omega$ we consider the local continuity modulus

$$\omega(f,x,h) = \omega_\Omega(f,x,h) = \sup_{\substack{z\in\Omega:\\ \varrho(x,z)\le h}} |f(x) - f(z)| \tag{2.26}$$

of a function f at the point x. Everywhere below we assume that $|h| < 1$. The function $\omega(f,x,h)$ is non-decreasing in h and tends to zero as $h \to +0$ for any continuous function on Ω and fixed x.

Lemma 2.10. *For all $x, y \in \Omega$ such that $\varrho(x,y) \le h$, the inequality*

$$\frac{1}{C}\omega(f,x,h) \le \omega(f,y,h) \le C\omega(f,x,h) \tag{2.27}$$

holds, where $C = [2k]+2$ and k is the constant from (2.2). *If $a(x) \in WL(\Omega)$, then*

$$\frac{1}{C}t^{a(x)} \le h^{a(y)} \le Ch^{a(x)} \tag{2.28}$$

for all x, y such that $\varrho(x,y) < h$, where $C \geq 1$ depends on the function a, but does not depend on x, y and h.

Proof. We have

$$\omega(f,y,h) = \sup_{z\in B(y,h)} |f(z)-f(y)| \leq \sup_{z\in B(y,h)} |f(z)-f(x)| + \omega(f,x,\varrho(x,y)).$$

It is easily seen that the condition $\varrho(x,y) \leq h$ implies the embedding $B(y,h) \subset B(x,2kh)$. Therefore,

$$\omega(f,y,h) \leq \omega(f,x,2kh) + \omega(f,x,h).$$

By the property $\omega(f,x,\lambda h) \leq ([\lambda]+1)\omega(f,x,h)$ of continuity moduli we arrive at the right-hand side of (2.27). Changing the roles of x and y, we obtain the left-hand-side one.

To prove(2.28), it suffices to observe that (2.28) is nothing else but $|a(x)-a(y)|\cdot|\ln t| \leq \ln C$ which follows from the WL-condition when $\varrho(x,y) \leq t$. □

Remark 2.11. Note that the moduli of continuity $\omega(f,x,t)$ satisfy the inequalities

$$\omega(f,x,h) \leq 2(1-\delta)\int_0^h \left(\frac{h}{t}\right)^{\delta} \frac{\omega(f,x,t)}{t}dt, \quad 0 < h \leq d, \tag{2.29}$$

$$\omega(f,x,h) \leq \frac{\beta}{1-2^{-\beta}}\int_h^2 \left(\frac{h}{t}\right)^{\beta} \frac{\omega(f,x,t)}{t}dt, \quad 0 < h \leq \frac{d}{2} \tag{2.30}$$

under any choice of $\delta < 1$ and $\beta > 0$. Inequality (2.30) is easily obtained by the estimation of the right-hand side from below by making use the monotonicity of the continuity modulus. Inequality (2.29) is similarly obtained by making use of the property

$$\frac{\omega(f,x,t)}{t} \geq \frac{1}{2}\frac{\omega(f,x,h)}{h}, \quad t < h \tag{2.31}$$

of continuity moduli.

In the sequel, the notation $\lambda(x)$ will always stand for a function $\lambda(x)$ on Ω satisfying the assumptions

$$\lambda_- := \inf_{x\in X} \lambda(x) > 0 \qquad \text{and} \quad \lambda_+ := \sup_{x\in X} \lambda(x) < 1.$$

Definition 2.12. *By $H^{\lambda(\cdot)}(\Omega)$ we denote the space of functions $f \in C(\Omega)$ such that*

$$\omega(f,x,h) \leq Ch^{\lambda(x)} \tag{2.32}$$

where $C > 0$ does not depend on $x, y \in \Omega$. Equipped with the norm

$$\|f\|_{H^{\lambda(\cdot)}(\Omega)} = \|f\|_{C(\Omega)} + \sup_{x\in\Omega}\sup_{h\in(0,1)} \frac{\omega(f,x,h)}{h^{\lambda(x)}},$$

this is a Banach space.

We will also deal with the generalized Hölder spaces $H^{w(\cdot,\cdot)}(\Omega)$ of functions whose continuity modulus is dominated by a given function $w(x,h)$, the case $w(x,h) = h^{\lambda(x)}$ being a particular case.

We denote $\mathbf{T} = \Omega \times [0,d]$. For a function $w(x,t)$ defined on $\mathbf{T}$ we introduce the bounds

$$w_-(t) = \inf_{x\in\Omega} w(x,t), \qquad \text{and} \qquad w_+(t) = \sup_{x\in\Omega} w(x,t).$$

Definition 2.13. *A function $w : \mathbf{T} \to \mathbb{R}^1_+$ is said to belong to the class $W = W(\mathbf{T})$, if*

1) $w(x,t)$ is continuous in $t \in [0,d]$ for every $x \in \Omega$,

2) $w_-(t) > 0$ when $t > 0$ and $\lim\limits_{t\to+0} w(x,t) = 0$ for every $x \in \Omega$,

3) $w(x,t)$ is almost increasing in t for every $x \in \Omega$.

Definition 2.14. *Let $w(x,h) \in W$. By $H^{w(\cdot)}(\Omega)$ we denote the space of functions $f \in C(\Omega)$ such that $\omega(f,x,h) \le cw(x,h)$, $x \in \Omega$ where $c > 0$ does not depend on x and h. Equipped with the norm*

$$\|f\|_{H^{w(\cdot)}(\Omega)} = \|f\|_{C(\Omega)} + \sup_{x\in\Omega,h>0} \frac{\omega(f,x,h)}{w(x,h)},$$

this is a Banach space.

2.4 On Zygmund-Bary-Stechkin classes $\Phi^{\delta(\cdot)}_{\beta(\cdot)}$

Definition 2.15. *We say that $w(x,t)$ belongs to a generalized Zygmund-Bary-Stechkin class $\Phi^{\delta(\cdot)}_{\beta(\cdot)} = \Phi^{\delta(\cdot)}_{\beta(\cdot)}(\mathbf{T})$, where $0 \le \delta(x) < \beta(x), x \in \Omega$, if $w(x,t) \in W$, and*

$$\int_0^h \left(\frac{h}{t}\right)^{\delta(x)} \frac{w(x,t)}{t}dt \le cw(x,h) \quad \text{and} \qquad \int_h^d \left(\frac{h}{t}\right)^{\beta(x)} \frac{w(x,t)}{t}dt \le cw(x,h), \tag{2.33}$$

where $0 < h < \frac{d}{2}$ and $c > 0$ does not depend on $h \in \left(0, \frac{d}{2}\right]$ and $x \in \Omega$. By $\Phi^{\delta(\cdot)}$ we also denote the corresponding class with only the first of the conditions in (2.33) satisfied, and by $\Phi_{\beta(\cdot)}$ the class with only the second one, so that $\Phi^{\delta(\cdot)}_{\beta(\cdot)} = \Phi^{\delta(\cdot)} \cap \Phi_{\beta(\cdot)}$.

From the definitions of the classes $\Phi_{\beta(\cdot)}$ and W it easily follows that a function $\omega(x,t) \in \Phi_{\beta(\cdot)}$ satisfies the property

$$\omega(x,t) \geq ct^{\beta(x)} \tag{2.34}$$

with a constant $c > 0$ not depending on x and t.

Such classes Φ^{δ}_{β} in the case of functions $w = w(t)$ and exponents β, δ, not depending on the parameter x, were introduced in the paper Bary-Stechkin [1] with $\delta = 0, \beta = 1, 2, 3, ...$; the classes Φ^{δ}_{β} with constant $0 \leq \delta < \beta$ appeared in [25]. We refer also to [8] for some properties of functions in these classes, see also their detailed study in [10].

We make use of the Matuszewska-Orlicz indices known in the theory of Orlicz spaces, see [14] and [13], of a function $\omega(x,t)$ with respect to the variable $t \in [0,d]$:

$$m(\omega,x) = \sup_{t>1} \frac{\ln \left[\underline{\lim}_{h\to 0} \frac{\omega(x,th)}{\omega(x,h)}\right]}{\ln\ t} = \lim_{t\to 0} \frac{\ln \left[\overline{\lim}_{h\to 0} \frac{\omega(x,th)}{\omega(x,h)}\right]}{\ln\ t} \tag{2.35}$$

$$M(\omega,x) = \inf_{t>1} \frac{\ln \left[\overline{\lim}_{h\to 0} \frac{\omega(x,th)}{\omega(x,h)}\right]}{\ln\ t} = \lim_{t\to \infty} \frac{\ln \left[\overline{\lim}_{h\to 0} \frac{\omega(x,th)}{\omega(x,h)}\right]}{\ln\ t} \tag{2.36}$$

depending on the parameter $x \in \Omega$, $m(\omega,x) \leq M(\omega,x)$. These indices in application to generalized Hölder spaces were studied in [15], [16], [18], [17], [20], [19], where in particular was shown that the belongness of a function $\omega(t)$ to Φ^{δ}_{β} with constant β and δ may be characterized in terms of the index numbers $m(w), M(w)$. In case of the class $\Phi^{\delta(\cdot)}_{\beta(\cdot)}$ depending on a parameter, a similar investigation was made in [21], including study of the uniformness of Zygmund type conditions (2.33), see Lemma 2.18. (In [21] the parameter x was a point of an arbitrary set).

We will also need the following numbers

$$m(\omega) = \sup_{r>1} \frac{\ln \left(\underline{\lim}_{h\to 0} \operatorname{ess\,inf}_{x\in\Omega} \frac{\omega(x,rh)}{\omega(x,h)}\right)}{\ln\ r}, \qquad M(\omega) = \inf_{r>1} \frac{\ln \left(\overline{\lim}_{h\to 0} \operatorname{ess\,sup}_{x\in\Omega} \frac{\omega(x,rh)}{\omega(x,h)}\right)}{\ln\ r}. \tag{2.37}$$

Note that $m(\omega) \leq \inf\limits_{x\in\mathbb{S}^{n-1}} m(\omega,x)$ and $M(\omega) \geq \sup\limits_{x\in\Omega} M(\omega,x)$.

Definition 2.16. By $\mathfrak{W}(\mathbf{T})$ we denote the subclass in $W(\mathbf{T})$ of functions of the form $\omega(x,t) = [\varphi(t)]^{\lambda(x)}$ where $\varphi \in W([0,d])$ and $\lambda \in L^{\infty}(\Omega), \operatorname{ess\,inf} \lambda(x) \geq 0$.

Lemma 2.17. *([21], Lemma 2.4) Let $w(x,t) = [\varphi(t)]^{\lambda(x)} \in \mathfrak{W}$. Then $m(\omega,x) = \lambda(x)m(\varphi)$, $M(\omega,x) = \lambda(x)M(\varphi)$, and*

$$m(\omega) = \inf_{x\in\Omega} m(\omega,x) \quad \textit{and} \quad M(\omega) = \sup_{x\in\Omega} M(\omega,x).$$

For the case $\beta(x) = \beta = const$ and $\delta(x) = \delta = const$, in [21] (Theorems 3.1 and 3.2) the following statement was proved.

Lemma 2.18. *Let* $\omega(x,t) \in \widetilde{W}(\mathbf{T})$. *Then* $\omega(x,t) \in \Phi^{\delta} \iff m(w) > \delta$, *and* $\omega(x,t) \in \Phi_{\beta} \iff M(w) < \beta$.

For the case of variable $\beta(x)$ and $\delta(x)$, the corresponding statement may be given in the following form obtained from Lemmas 2.18 and 2.17.

Corollary 2.19. *Let* $\omega_{\delta} = \frac{\omega(x,t)}{t^{\delta(x)}}$ *and* $\omega_{\beta} = \frac{\omega(x,t)}{t^{\beta(x)}}$. *Then*

$$\omega(x,t) \in \Phi^{\delta(\cdot)} \iff m(\omega_{\delta}) > 0, \tag{2.38}$$

$$\omega(x,t) \in \Phi_{\beta(\cdot)} \iff M(w_{\beta}) < 0. \tag{2.39}$$

In case of functions $\omega(x,t) \in \mathfrak{W}(\mathbf{T})$, *the equivalencies* (2.38), (2.39) *take the form*

$$\omega(x,t) \in \Phi^{\delta(\cdot)} \iff \operatorname*{ess\,inf}_{x\in\Omega}[m(w,x) - \delta(x)] > 0, \tag{2.40}$$

$$\omega(x,t) \in \Phi_{\beta(\cdot)} \iff \operatorname*{ess\,sup}_{x\in\Omega}[M(w,x) - \beta(x)] < 0. \tag{2.41}$$

We will make use of the following property of the bounds for functions $\omega(x,t) \in W(\mathbf{T})$ in terms of their indices:

$$c_1 t^{M(w)+\varepsilon} \le \omega(x,t) \le c_2 t^{m(\omega)-\varepsilon}, \qquad 0 \le t \le d \tag{2.42}$$

where $\varepsilon > 0$ and the constants c_1, c_2 may depend on ε, but do not depend on x and t (see [21], Theorem 3.5).

3 Potentials and hypersingular integrals of variable order in the space $H^{\lambda(\cdot)}(\Omega)$.

Everywhere in the sequel we suppose that $\varrho(x,y)$ is either a metric or a regular quasidistance of order $\theta \in (0,1]$.

3.1 Zygmund type estimates of potentials.

We assume that $\alpha \in C(\Omega)$ and $\Re\alpha \in WL(\Omega)$.

Remark 3.1. If $\Re\alpha \in WL(\Omega)$, then

$$e^{-A}t^{\Re\alpha(x)} \le t^{\Re\alpha(y)} \le e^{A}t^{\Re\alpha(x)} \quad \text{for} \quad \varrho(x,y) \le \min\left(t, \frac{1}{2}\right), \tag{3.1}$$

where $A = A(\Re\alpha)$ is the constant from (2.5) for the function $a(x) = \Re\alpha(x)$.

It is clear that in Hölder norm estimations of functions $I^{\alpha}f$, the case $f \equiv const$ plays an important role, in the case where

$$\mathfrak{I}_{\alpha}(x) := I^{\alpha}(1)(x) = \int\limits_{\Omega} \frac{d\mu(z)}{\varrho(x,z)^{N-\alpha(x)}} \tag{3.2}$$

is well defined. Observe that in the Euclidean case $\Omega = X = \mathbb{R}^N$, this integral although not well directly defined, may be treated as a constant in the case $\alpha(x) = \alpha = const$ in the sense that the cancellation property

$$\int\limits_{\mathbb{R}^N} \left[\frac{1}{|z-x|^{N-\alpha}} - \frac{1}{|z-y|^{N-\alpha}}\right] dz \equiv 0, \quad 0 < \Re\alpha < 1, \quad x, y \in \mathbb{R}^N \tag{3.3}$$

holds. For constant α, the function $\mathfrak{I}_{\alpha}(x)$ is also constant in the case $\Omega = X = \mathbb{S}^{N-1}$, which fails when $\alpha = \alpha(x)$ and the cancellation property of the type

$$\int\limits_{\Omega} \left[\frac{1}{|z-x|^{N-\alpha(x)}} - \frac{1}{|z-y|^{N-\alpha(y)}}\right] d\mu(z) \equiv 0, \tag{3.4}$$

no more holds even for $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{S}^{N-1}$ (see for instance [5] on the importance of the cancellation property $I^{\alpha}(1) \equiv const$ for the validity of mapping properties of potentials within Hölder spaces on quasimetric measure spaces)

When considering Hölder type spaces $H^{\lambda(\cdot)}(\Omega)$ which contain constants, the condition

$$\mathfrak{I}_{\alpha} \in H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$$

is necessary for the mapping

$$I^{\alpha} : H^{\lambda(\cdot)}(\Omega) \to H^{\lambda(\cdot)+\alpha(\cdot)}(\Omega)$$

to hold.

Remark 3.2. Let $\inf\limits_{x\in\Omega} \Re\alpha(x) \ge 0$ and $x, y \notin \Pi_{\alpha}$. Then

$$|\mathfrak{I}_{\alpha}(x) - \mathfrak{I}_{\alpha}(y)| \le C\frac{|\alpha(x)-\alpha(y)|}{\min(\Re\alpha(x), \Re\alpha(y))} + \left|\int\limits_{\Omega} \left[\varrho(x,z)^{\alpha(x)-N} - \varrho(y,z)^{\alpha(x)-N}\right] d\mu(z)\right| \tag{3.5}$$

and

$$|\alpha(x)\mathfrak{I}_\alpha(x)-\alpha(y)\mathfrak{I}_\alpha(y)|\le C\,|\alpha(x)-\alpha(y)| \tag{3.6}$$
$$+\min(\Re\alpha(x),\Re\alpha(y))\left|\int\limits_\Omega\left[\varrho(x,z)^{\alpha(x)-N}-\varrho(y,z)^{\alpha(x)-N}\right]d\mu(z)\right|$$

where $C>0$ does not depend on $x,y\in\Omega$.

Proof. We have

$$\mathfrak{I}_\alpha(x)-\mathfrak{I}_\alpha(y)=\int\limits_\Omega\left[\varrho(y,z)^{\alpha(x)-N}-\varrho(y,z)^{\alpha(y)-N}\right]d\mu(z)+\int\limits_\Omega\left[\varrho(x,z)^{\alpha(x)-N}-\varrho(y,z)^{\alpha(x)-N}\right]d\mu(z). \tag{3.7}$$

By (4.33) with $f(t)=\varrho^t$, after easy estimations we obtain

$$\left|\varrho^{\alpha(x)-N}-\varrho^{\alpha(y)-N}\right|\le|\alpha(x)-\alpha(y)|\varrho^{\min(\Re\alpha(x),\Re\alpha(y))-N}|\ln\varrho|,\quad 0<\varrho\le d<\infty$$

which yields (3.5) after easy calculations with estimate (2.14) taken into account. Estimate (3.6) easily follows from (3.5). □

Remark 3.3. The meaning of estimates (3.5)-(3.6) is in the fact that the second term on the right-hand sides may be subject to the cancellation property: at the least it disappears when $\Omega=X=\mathbb{R}^N$ or $\Omega=X=\mathbb{S}^{N-1}$.

The estimate (3.9) provided by the following theorem clearly shows the worsening of the behaviour of the local continuity modulus $\omega(I^\alpha f,x,h)$ when x approaches the points where $\alpha(x)$ vanishes. We also give a weighted estimate exactly with the weight $\alpha(x)$. For the latter we exclude purely imaginary orders $\alpha(x)=i\vartheta(x)$ by the following condition

$$\max_{x\in\Omega}|\arg\alpha(x)|<\frac{\pi}{2}-\varepsilon,\quad \varepsilon>0. \tag{3.8}$$

We use the notation

$$\alpha_h(x)=\min_{\varrho(x,y)<h}\Re\alpha(y).$$

Theorem 3.4. *Let Ω be a bounded open set in X, $\alpha\in C(\Omega)$ and $\Re\alpha\in WL(\Omega)$ and $0\le\inf\limits_{x\in\Omega}\Re\alpha(x)\le\sup\limits_{x\in\Omega}\Re\alpha(x)<1$, and let $0<h<\lambda d$, $\lambda=\min\left(\frac12,e^{-\frac1N}\right)$. Then for all the points $x\in\Omega\backslash\Pi_\alpha$ such that $\alpha_h(x)\ne0$, the following Zygmund type estimate is valid*

$$\omega(I^\alpha f,x,h)\le\frac{C}{\alpha_h(x)}h^{\Re\alpha(x)}\omega(f,x,h)+Ch^\theta\int\limits_h^d\frac{\omega(f,x,t)dt}{t^{1+\theta-\Re\alpha(x)}} \tag{3.9}$$

$$+C\omega(\alpha,x,h)\int\limits_h^d \frac{\omega(f,x,t)dt}{t^{2-\Re\alpha(x)}} + C\omega(\mathfrak{I}_\alpha,x,h)\|f\|_{C(\Omega)},$$

where the constant $C>0$ does not depend on f,x and h.

If additionally $\alpha(x)$ satisfies condition (3.8)*, then for all the points $x\in\Omega\backslash\Pi_\alpha$ the weighted estimate holds*

$$\omega(\alpha I^\alpha f,x,h) \le Ch^{\Re\alpha(x)}\omega(f,x,h) + Ch^\theta \int\limits_h^d \frac{\omega(f,x,t)dt}{t^{1+\theta-\Re\alpha(x)}} \tag{3.10}$$

$$+C\omega(\alpha,x,h)\int\limits_h^d \frac{\omega(f,x,t)dt}{t^{2-\Re\alpha(x)}} + C\omega(\alpha\mathfrak{I}_\alpha,x,h)\|f\|_{C(\Omega)},$$

Proof. Given $x,y\in\Omega$, we represent the difference $(I^\alpha f)(x)-(I^\alpha f)(y)$ in the following form (compare with similar representations in [26], [27], [35] in the case of $X=\mathbb{S}^{N-1}$)

$$(I^\alpha f)(x)-(I^\alpha f)(y) = \tag{3.11}$$

$$= \int\limits_{\varrho(x,z)<2h} [f(z)-f(x)]\varrho(x,z)^{\alpha(x)-n}d\mu(z) - \int\limits_{\varrho(x,z)<2h} [f(z)-f(x)]\varrho(y,z)^{\alpha(y)-n}d\mu(z)+$$

$$+ \int\limits_{\varrho(x,z)>2h} [f(z)-f(x)]\left\{\varrho(x,z)^{\alpha(y)-N} - \varrho(y,z)^{\alpha(y)-N}\right\}d\mu(z)+$$

$$+ \int\limits_{\varrho(x,z)>2h} [f(z)-f(x)]\left\{\varrho(x,z)^{\alpha(x)-n} - \varrho(x,z)^{\alpha(y)-n}\right\}d\mu(z)$$

$$+f(x)\int\limits_\Omega \left\{\varrho(x,z)^{\alpha(x)-N} - \varrho(y,z)^{\alpha(y)-N}\right\}d\mu(z) = I_1+I_2+I_3+I_4+I_5.$$

For I_1, we have

$$|I_1| \le \omega(f,x,2h)\int\limits_{\varrho(x,z)<2h} \frac{d\mu(z)}{\varrho(x,z)^{N-\Re\alpha(x)}}.$$

By Corollary 2.7 and the property $\omega(f,x,2h)\le 2\omega(f,x,h)$ of the continuity modulus, we get

$$|I_1| \le 4C(K,N)\frac{h^{\Re\alpha(x)}\omega(f,x,h)}{\Re\alpha(x)}, \tag{3.12}$$

where $C(K,N)=eK(N+1)$ The term I_2 is similarly estimated, since

$$\{z:\varrho(x,z)<2h\}\subset\{z:\varrho(y,z)<3kh\}, \tag{3.13}$$

where k is the constant from (2.2) and we obtain

$$|I_2| \le 12kC(K,N)\frac{h^{\Re\alpha(x)}\omega(f,x,h)}{\Re\alpha(y)} \le 12kC(K,N)\frac{h^{\Re\alpha(x)}\omega(f,x,h)}{\alpha_h(x)}. \tag{3.14}$$

To estimate I_3, we make use of Lemma 2.9 and obtain

$$|I_3| \le C_1(N,M,\alpha)h^\theta \int\limits_{\varrho(x,z)>2h} \frac{\omega(f,x,\varrho(x,z))}{\varrho(x,z)^{N+\theta-\Re\alpha(y)}}d\mu(z), \tag{3.15}$$

where $\theta = 1$ when ϱ is a metric and $0 < \theta \le 1$ when (X,ϱ,μ) is regular of order $\theta \in (0,1]$, $C_1(N,M,\alpha) = 12M\sup_{y\in\Omega}|N-\alpha(y)|$ does not depend on, x, y, h and M is the constant from (2.3). The integral on the right-hand sides of (3.15) is estimated by means of inequality (2.21) of Lemma 2.8:

$$|I_3| \le C_2(K,N,M,\alpha)h^\theta \int\limits_h^d \frac{\omega(f,x,t)}{t^{1+\theta-\Re\alpha(y)}}dt$$

with $C_2(K,N,M,\alpha) = 4eK(N+1)C_1(N,M,\alpha)$.

For I_4 we have

$$|I_4| \le \int\limits_{\varrho(x,z)>2h} \frac{\omega(f,x,\varrho(x,z))}{\varrho(x,z)^{N-\Re\alpha(y)}}\left|\varrho(x,z)^{\alpha(x)-\alpha(y)}-1\right|d\mu(z).$$

By (2.24) with $a = \varrho = \varrho(x,z)$ and $b = 1$ we have

$$\left|\varrho^{\alpha(x)-\alpha(y)}-1\right| \le (d+1)\frac{|\alpha(x)-\alpha(y)|}{[\min\{1,\varrho\}]^{\Re\alpha(y)-\Re\alpha(x)+1}} \le \frac{C_0|\alpha(x)-\alpha(y)|}{\varrho^{\Re\alpha(y)-\Re\alpha(x)+1}}, \tag{3.16}$$

for all $0 < \varrho \le d < \infty$, where C_0 depends only on d and $\max_{x,y\in\Omega}|\Re\alpha(y)-\Re\alpha(x)|$, but do not depend on $x, y \in \Omega$ and $\varrho \in (0,d]$. Therefore,

$$|I_4| \le C_0|\alpha(x)-\alpha(y)| \int\limits_{\varrho(x,z)>2h} \frac{\omega(f,x,\varrho(x,z))}{\varrho(x,z)^{N+1-\Re\alpha(x)}}d\mu(z)$$

and then

$$|I_4| \le C_4\omega(\alpha,x,h)\int\limits_h^d \frac{\omega(f,x,t)}{t^{2-\Re\alpha(x)}}dt$$

by inequality (2.21) of Lemma 2.8, where $C_4 = 4C_0C(K,N)$ does not depend on x, y, h.

Gathering the estimates for $I_k, k = 1,2,3,4$, and taking into account that $|I_5| \le |f(x)|\omega(\mathfrak{I}_\alpha,x,h)$, we get at (3.9).

To get at (3.10), we make use of the representation

$$\alpha(x)(I^\alpha f)(x)-\alpha(y)(I^\alpha f)(y)= \tag{3.17}$$

$$=\alpha(x)\int\limits_{\varrho(x,z)<2h}[f(z)-f(x)]\varrho(x,z)^{\alpha(x)-n}d\mu(z)-\alpha(y)\int\limits_{\varrho(x,z)<2h}[f(z)-f(x)]\varrho(y,z)^{\alpha(y)-n}d\mu(z)+$$

$$+\alpha(y)\int\limits_{\varrho(x,z)>2h}[f(z)-f(x)]\left\{\varrho(x,z)^{\alpha(y)-N}-\varrho(y,z)^{\alpha(y)-N}\right\}d\mu(z)+$$

$$+\alpha(x)\int\limits_{\varrho(x,z)>2h}[f(z)-f(x)]\left\{\varrho(x,z)^{\alpha(x)-n}-\varrho(x,z)^{\alpha(y)-n}\right\}d\mu(z)$$

$$+f(x)\int\limits_{\Omega}\left\{\alpha(x)\varrho(x,z)^{\alpha(x)-N}-\alpha(y)\varrho(y,z)^{\alpha(y)-N}\right\}d\mu(z)$$

$$+[\alpha(x)-\alpha(y)]\int\limits_{\varrho(x,z)>2h}[f(z)-f(x)]\varrho(x,z)^{\alpha(y)-n}d\mu(z)=I_1^\alpha+I_2^\alpha+I_3^\alpha+I_4^\alpha+I_5^\alpha+I_6^\alpha.$$

Estimations of the terms $I_k^\alpha, k=1,..,5$, follow the same line as those for the terms $I_k, k=1,..,5$, above, while $|I_6^\alpha|\le C\omega(\alpha\mathfrak{J},x,h)\int\limits_h^d\frac{\omega(f,x,t)}{t^{2-\Re\alpha(x)}}$ by Lemma 2.8 and Remark 3.1. After collecting the estimates we arrive at (3.10) with (3.8) taken into account. □

3.2 Zygmund type estimates of hypersingular integrals

Remark 3.5. Note that the second term on the right-hand side of estimate (3.18) proved in the following theorem is taken in the form not symmetric with respect to x,y (compare with the first term), because all the second term calculated at the point x is equivalent to that calculated at the point y according to Lemma 2.28, due to the integration over $\{t>h\}$.

Theorem 3.6. *Let $\alpha\in C(\Omega),\Re\alpha\in WL(\Omega)$ and $0\le\min\limits_{x\in\Omega}\Re\alpha(x)\le\max\limits_{x\in\Omega}\Re\alpha(x)<1$. If $f\in C(\Omega)$, then for all $x,y\in\Omega$ with $\varrho(x,y)<h$ such that $\Re\alpha(x)\ne0$ and $\Re\alpha(y)\ne0$, the following estimate is valid*

$$|(D^\alpha f)(x)-(D^\alpha f)(y)|\le\frac{C}{\min(\Re\alpha(x),\Re\alpha(y))}\int\limits_0^h\left[\frac{\omega(f,x,t)}{t^{1+\Re\alpha(x)}}+\frac{\omega(f,y,t)}{t^{1+\Re\alpha(y)}}\right]dt \tag{3.18}$$

$$+C\int\limits_h^2\left[\omega(\alpha,x,h)+h^\theta t^{1-\theta}\right]\frac{\omega(f,x,t)dt}{t^{2+\Re\alpha(x)}},$$

where $C>0$ does not depend on x, y and h.

Proof. We represent the difference $(D^\alpha f)(x)-(D^\alpha f)(y)$ as

$$(D^\alpha f)(x)-(D^\alpha f)(y)=A_1+A_2+A_3+A_4+A_5,$$

where

$$A_1=\int\limits_{\varrho(x,z)<2h}[f(z)-f(x)]\varrho(x,z)^{-N-\alpha(x)}d\mu(z),\quad A_2=\int\limits_{\varrho(x,z)<2h}[f(y)-f(z)]\varrho(y,z)^{-N-\alpha(y)}d\mu(z),$$

$$A_3=\int\limits_{\varrho(x,z)>2h}[f(z)-f(x)]\left\{\varrho(x,z)^{-N-\alpha(y)}-\varrho(y,z)^{-N-\alpha(y)}\right\}d\mu(z),$$

$$A_4=[f(y)-f(x)]\int\limits_{\varrho(x,z)>2h}\frac{d\mu(z)}{\varrho(y,z)^{N+\alpha(y)}},$$

$$A_5=\int\limits_{\varrho(x,z)>2h}[f(z)-f(x)]\left\{\varrho(x,z)^{-N-\alpha(x)}-\varrho(x,z)^{-N-\alpha(y)}\right\}d\mu(z). \tag{3.19}$$

Estimation of the terms $A_k, k=1,...,5$, follows more or less the same lines as in the proof of estimate (3.9). Thus for A_1 by Lemma 2.5 we obtain

$$|A_1|\le\int\limits_{\varrho(x,z)<2h}\frac{\omega(f,x,\varrho(x,z))}{\varrho(x,z)^{N+\Re\alpha(x)}}d\mu(z)\le C\int\limits_0^{2h}\frac{\omega(f,x,t)}{t^{1+\Re\alpha(x)}}dt. \tag{3.20}$$

For A_2, by (3.13) and the same Lemma 2.5 we have

$$|A_2|\le\int\limits_{\varrho(y,z)<3kh}\frac{\omega(f,y,\varrho(y,z))}{\varrho(y,z)^{N+\Re\alpha(y)}}d\mu(z)$$

$$\le C\int\limits_0^{3kh}\frac{\omega(f,y,t)}{t^{1+\Re\alpha(y)}}dt\le C\int\limits_0^{h}\frac{\omega(f,y,t)}{t^{1+\Re\alpha(y)}}dt,$$

where we have used the property (2.31). In the case of A_3, we make use of (2.22) and (2.23) and get

$$|A_3|\le Ch^\theta\int\limits_{\varrho(x,z)>2h}\frac{\omega(f,x,\varrho(x,z))}{\varrho(x,z)^{N+\Re\alpha(y)+\theta}}d\mu(z),$$

where $C=M2^{N+2}3^{1-\theta}\cdot\max_{y\in\Omega}|N-al(y)|$. Then by (2.20), we obtain

$$|A_3|\le Ch^\theta\int\limits_h^d\frac{\omega(f,x,t)}{t^{1+\theta+\Re\alpha(y)}}dt.$$

For A_4 we observe that $\{z \in \Omega : \varrho(x,z) > 2h\} \subset \{z \in \Omega : \varrho(y,z) > \frac{h}{k}$ and use (2.20) again, which yields

$$|A_4| \le \omega(f,y,h) \int\limits_{\varrho(y,z)>\frac{h}{k}} \frac{d\mu(z)}{\varrho(y,z)^{N+\Re\alpha(y)}} \le C\omega(f,y,h) \int\limits_{\frac{h}{k}}^{d} \frac{dt}{t^{1+\Re\alpha(y)}} \le \frac{C}{\Re\alpha(y)} \frac{\omega(f,y,h)}{h^{\Re\alpha(y)}}.$$

Then by (2.29) we arrive at the estimate

$$|A_4| \le \frac{C}{\Re\alpha(y)} \int\limits_0^h \frac{\omega(f,y,t)}{t^{1+\Re\alpha(y)}} dt.$$

Finally, for A_5, by (3.16) we obtain

$$|A_5| \le \omega(\alpha,x,h) \int\limits_{\varrho(x,y)>2h} \frac{\omega(f,x,\varrho(x,z))}{\varrho(x,z)^{N+1+\Re\alpha(x)}} d\mu(z)$$

and then by (2.20)

$$|A_5| \le C\omega(\alpha,x,h) \int\limits_h^d \frac{\omega(f,x,t)}{t^{2+\Re\alpha(x)}} dt. \tag{3.21}$$

Observe that the bounds for $|A_3|$ and $|A_4|$ are dominated by the bounds for $|A_5|$, because $h \le C\omega(\alpha,x,h)$ with the constant $C > 0$ not depending on x, if $\alpha(x)$ is not an identical constant. The latter follows from the almost monotonicity property $\frac{\omega(\alpha,x,h)}{h} \ge 2\frac{\omega(\alpha,x,d)}{d}$ and the fact that $\inf\limits_{x\in\Omega} \omega(\alpha,x,d) > 0$ for any continuous function $\alpha(x)$ different from a constant.

Gathering all the estimates for $A_1, ..., A_5$, we arrive at (3.18). □

Remark 3.7. Similarly to Theorem 3.4, it is possible to obtain weighted estimates for $D^\alpha f$ with the weight $\alpha(x)$. We do not dwell on such estimations in this paper.

3.3 Theorems on mapping properties for potentials and hypersingular operators of variable order in the spaces $H^{w(\cdot)}(\Omega)$.

Recall that for the potential operator I^α we allow the variable order $\alpha(x)$ to have a degenerate $\Re\alpha(x)$ on a set Π_α (of measure zero). We consider the weighted space

$$H^{\omega_\alpha}(\Omega,\alpha) = \{f : \alpha(x)f(x) \in H^{\omega_\alpha(\cdot)}(\Omega\},$$

where

$$\omega_\alpha(x,t) = t^{\Re\alpha(x)} w(x,t).$$

Theorem 3.8. *Let*

$$\alpha(x) \in \mathrm{Lip}(\Omega) \quad \textit{and} \quad 0 \le \Re\alpha(x), \ \max_{x\in\Omega} \Re\alpha(x) < \theta, \tag{3.22}$$

let $\mu\{x : \Re\alpha(x) = 0\} = 0$ *and let condition* (3.8) *be satisfied, and*

$$w(x,t) \in \Phi_{\theta-\Re\alpha(x)}. \tag{3.23}$$

If

$$\alpha\mathfrak{I}_\alpha \in H^{\omega_\alpha(\cdot)}, \tag{3.24}$$

then the operator I^α *is bounded from the space* $H^{w(\cdot)}(\Omega)$ *into the weighted space* $H^{w_\alpha(\cdot)}(\Omega,\alpha)$.

Proof. It suffices to show that

$$\frac{\omega(\alpha I^\alpha f,h,x)}{h^{\Re\alpha(x)}w(x,h)} \le c\|f\|_{H^{w(\cdot)}} \quad \text{for} \quad f \in H^{w(\cdot)}(\Omega) \tag{3.25}$$

for small $h > 0$. Under the assumptions of the theorem, from (3.10) we have

$$\omega\left(\alpha I^{\alpha(\cdot)}f,x,h\right) \le c\left(h^{\Re\alpha(x)}\omega(x,h) + \int_h^d \left[\left(\frac{h}{t}\right)^\theta + \frac{h}{t}\right]\frac{\omega(x,t)dt}{t^{1-\Re\alpha(x)}}\right)\|f\|_{H^{w(\cdot)}} \tag{3.26}$$

$$\le c\left(h^{\Re\alpha(x)}\omega(x,h) + h^\theta\int_h^d \frac{\omega(x,t)dt}{t^{1+\theta-\Re\alpha(x)}}\right)\|f\|_{H^{w(\cdot)}}$$

By condition (3.23), the integral term on the right-hand side is dominated by $Ch^{\Re\alpha(x)}\omega(x,h)$. Therefore, (3.26) yields (3.25). □

A "non-degeneracy" version of Theorem 3.8, obtained similarly from (3.9), runs as follows.

Theorem 3.9. *Let*

$$\alpha \in \mathrm{Lip}(\Omega), \quad 0 < \min_{x\in\Omega} \Re\alpha(x) \le \max_{x\in\Omega} \Re\alpha(x) < \theta. \tag{3.27}$$

Under conditions (3.23) *and* (3.24)*, the operator* I^α *is bounded from the space* $H^{w(\cdot)}(\Omega)$ *into the space* $H^{w_\alpha(\cdot)}(\Omega)$.

We also reformulate Theorems 3.8 and 3.9, replacing the information about the belongness of $\omega(x,h)$ to the Zygmund-Bary-Stechkin class $\Phi_{\theta-\Re\alpha(\cdot)}$ by the direct inequalities

imposed on the index numbers $m(\omega,x)$ and $M(\omega,x)$ of $\omega(x,h)$, which is possible by Corollary 2.19. To this end, we will use the condition

$$M(\omega_\alpha)<\theta \tag{3.28}$$

which takes the form

$$\sup_{x\in\Omega}[M(\omega,x)+\Re\alpha(x)]<\theta. \tag{3.29}$$

in the case where $\omega(x,t)\in\mathfrak{W}$.

Theorem 3.10. *Let $\omega\in W(\mathbf{T})$ and conditions* (3.24) *and* (3.28) *be satisfied.*
I. Under conditions (3.8) *and* (3.22) *the operator I^α is bounded from the space $H^{\omega(\cdot)}(\Omega)$ into the weighted space $H^{\omega_\alpha(\cdot)}(\Omega,\alpha)$.*
II. Under condition (3.27)*, the operator I^α is bounded from the space $H^{\omega(\cdot)}(\Omega)$ into the space $H^{\omega_\alpha(\cdot)}(\Omega)$*

Proof. The statements of the theorem follow as a direct reformulation of Theorems 3.8 and 3.9 via Corollary 2.19. □

Remark 3.11. In the case of "variable order Hölder space", that is, $\omega(x,t)=t^{\lambda(x)}$, condition (3.28)-(3.29) reduces to

$$\sup_{x\in\Omega}[\lambda(x)+\Re\alpha(x)]<\theta.$$

In the following theorem we use the notation

$$\omega_{-\alpha}(x,t)=t^{-\Re\alpha(x)}\omega(x,t)\quad\text{and}\quad\widetilde{\omega}_{-\alpha}(x,h)=\sup_{y:|y-x|<h}\omega_{-\alpha}(y,h).$$

Theorem 3.12. *Let conditions*

$$\alpha\in\mathrm{Lip}(\Omega),\qquad 0<\min_{x\in\Omega}\Re\alpha(x)\le\max_{x\in\Omega}\Re\alpha(x)<1 \tag{3.30}$$

be fulfilled. The operator $D^{\alpha(\cdot)}$ is bounded from the space $H^{\omega(\cdot)}(\Omega)$ into the space $H^{\widetilde{\omega}_{-\alpha}(\cdot)}(\Omega)$, if

$$\widetilde{\omega}(x,t)\in\Phi^{\Re\alpha(x)}_{\theta+\Re\alpha(x)},$$

or equivalently

$$m(\widetilde{w}_{-\alpha})>0\quad\textit{and}\quad M(\widetilde{\omega}_{-\alpha})<\theta; \tag{3.31}$$

in particular, when $\omega(x,t)\in\mathfrak{W}(\mathbf{T})$, conditions (3.31) *take the form*

$$0<\operatorname*{ess\,inf}_{x\in\Omega}\{m(\omega,x)-\Re\alpha(x)\},\quad\operatorname*{ess\,sup}_{x\in\Omega}[M(\omega,x)-\Re\alpha(x)]<\theta. \tag{3.32}$$

In the case of "variable order Hölder space" with $\omega(x,t) = t^{\lambda(x)}$*, one should take* $m(\omega,x) = M(\omega,x) = \lambda(x)$.

Proof. The proof of Theorem 3.12 is obtained similarly to that of Theorem 3.8, by means of the Zygmund type estimate (3.18). □

4 Appendix. Proof of inequality (2.24)

Since γ is complex, one may not use the mean value theorem in the Lagrange form, but its integral form

$$f(b) - f(a) = (b-a)\int_0^1 f'(a + s(b-a))ds \tag{4.33}$$

serves well for complex-valued functions $f(t)$. For $f(t) = t^{-\gamma}$ we arrive at

$$|a^{-\gamma} - b^{-\gamma}| \leq |\gamma||a-b|\int_0^1 [a + s(b-a)]^{-\Re\gamma - 1} ds,$$

from which (2.24) easily follows.

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References

[1] N.K. Bary and S.B. Stechkin. Best approximations and differential properties of two conjugate functions (in Russian). *Proceedings of Moscow Math. Soc.*, 5:483–522, 1956.

[2] L. Diening, P. Hästö, and A. Nekvinda. Open problems in variable exponent Lebesgue and Sobolev spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 - June 2, 2004*. Math. Inst. Acad. Sci. Czech Republick, Praha.

[3] D.E. Edmunds, V. Kokilashvili, and A. Meskhi. *Bounded and compact integral operators*, volume 543 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2002.

[4] J. García-Cuerva and A.E. Gatto. Boundedness properties of fractional integral operators associated to non-doubling measures. *Studia Math.*, 162:245–261, 2004.

[5] A.E. Gatto. On fractional calculus associated to doubling and non-doubling measures. In J. Marshall (ed.) et al. Ash, editor, *Harmonic analysis. Calderon-Zygmund and beyond. A conference in honor of Stephen Vagi's retirement, Chicago, IL, USA, December 6–8, 2002*, volume 411, pages 15–37. Providence, RI: American Mathematical Society, 2006. Contemporary Mathematics.

[6] A.E. Gatto, C. Segovia, and S. Vagi. On fractional differentiation on spaces of homogeneous type. *Revista Mat. Iberoamer*, 12(1), 1996. 1-35.

[7] A.E. Gatto and S. Vagi. Fractional integrals on spaces of homogeneous type. In Cora Sadosky (ed.), editor, *Analysis and Partial Differential Equations*, volume 122, pages 171–216. Marcel Dekker, New York and Basel, 1990. Lecture Notes in Pure and Appl. Math.

[8] A.I. Guseinov and H.Sh. Mukhtarov. *Introduction to the theory of nonlinear singular integral equations (in Russian)*. Moscow, Nauka, 1980. 416 pages.

[9] M.G. Hajibayev and S.G. Samko. Generalized potentials in variable exponent Lebesgue spaces on homogeneous spaces. *Math. Nachr.*, to appear

[10] N.K. Karapetiants and N.G. Samko. Weighted theorems on fractional integrals in the generalized Hölder spaces $H_0^\omega(\rho)$ via the indices m_ω and M_ω. *Fract. Calc. Appl. Anal.*, 7(4):437–458, 2004.

[11] V. Kokilashvili. On a progress in the theory of integral operators in weighted Banach function spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 - June 2, 2004*. Math. Inst. Acad. Sci. Czech Republick, Praha.

[12] R. A. Macías and C. Segovia. Lipshitz functions on spaces of homogeneous type. *Adv. Math.*, 33:257–270, 1979.

[13] L. Maligranda. *Orlicz spaces and interpolation.* Departamento de Matemática, Universidade Estadual de Campinas, 1989. Campinas SP Brazil.

[14] W. Matuszewska and W. Orlicz. On some classes of functions with regard to their orders of growth. *Studia Math.*, 26:11–24, 1965.

[15] N.G. Samko. Singular integral operators in weighted spaces with generalized Hölder condition. *Proc. A. Razmadze Math. Inst*, 120:107–134, 1999.

[16] N.G. Samko. Criterion of Fredholmness of singular operators with piece-wise continuous coefficients in the generalized Hölder spaces with weight. In *Proceedings of IWOTA 2000, Setembro 12-15, Faro, Portugal*, pages 363–376. Birkhäuser, In: "Operator Theory: Advances and Applications", v. 142, 2002.

[17] N.G. Samko. On compactness of Integral Operators with a Generalized Weak Singularity in Weighted Spaces of Continuous Functions with a Given Continuity Modulus. *Proc. A. Razmadze Math. Inst*, 136:91 – 113, 2004.

[18] N.G. Samko. On non-equilibrated almost monotonic functions of the Zygmund-Bary-Stechkin class. *Real Anal. Exch.*, 30(2):727, 2005.

[19] N.G. Samko. Singular integral operators in weighted spaces of continuous functions with an oscillating continuity modulus and oscillating weights. *Operator Theory: Advances and Applications, Birkhäuser, Proc. of the conference IWOTA, Newcastle, July 2004*, 171:323–347, 2006.

[20] N.G. Samko. Singular integral operators in weighted spaces of continuous functions with non-equilibrated continuity modulus. *Mathem. Nachrichten*, 279(12):1359–1375, 2006.

[21] N.G. Samko. Zygmund-Bary-Stechkin classes with parameter. *J. Funct. Spaces Appl.*, 2008. to appear.

[22] S.G. Samko. Fractional integration and differentiation of variable order. *Anal. Math.*, 21(3):213–236, 1995.

[23] S.G. Samko. *Hypersingular Integrals and their Applications.* London-New-York: Taylor & Francis, Series *"Analytical Methods and Special Functions*", vol. 5, 2002. 358 + xvii pages.

[24] S.G. Samko. On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. *Integr. Transf. and Spec. Funct*, 16(5-6):461–482, 2005.

[25] S.G. Samko and Kh.M. Murdaev. Weighted Zygmund estimates for fractional differentiation and integration, and their applications. *Trudy Matem. Inst. Steklov*, 180:197–198, 1987. English transl. in Proc. Steklov Inst. Math., AMS, 1989, issue 3, 233-235.

[26] B.G. Vakulov. Potential type operator on a sphere in generalized Hölder classes. *Izv. Vuzov. Matematika*, (11):66–69, 1986. English transl.: Soviet Math. (Iz. VUZ) 30 (1986), no. 11, 90–94.

[27] B.G. Vakulov. Spherical operators of potential type in generalized weighted Hölder space on a sphere (in Russian). *Izv. Vyssh. Uchebn. Zaved. Sev.-Kavk. Reg. Estestv. Nauki*, (4):5–10, 127, 1999.

[28] B.G. Vakulov. Spherical potentials in weighted Hölder spaces of variable order. *Dokl. Akad. Nauk*, 400(1):7–10, 2005. Transl. in Doklady Mathematics, 2005, 71(1): 1-4.

[29] B.G. Vakulov. Spherical potentials of complex order in the variable order Holder spaces. *Integral Transforms Spec. Funct.*, 16(5-6):489–497, 2005.

[30] B.G. Vakulov. Spherical convolution operators in Hölder spaces of variable order (in Russian). *Matem. Zametki*, 80(5):683–695, 2006. Transl. in Math. Notes, 80:5, 2006, 645–657.

[31] B.G. Vakulov. Spherical potentials of complex order in the spaces with generalized variable höolder condition (in Russian). *Dokl. Akad. Nauk*, 407(1):12–15, 2006. Transl. in Doklady Mathematics, 2006, 73(2): 165168.

[32] B.G. Vakulov, N.K. Karapetiants, and L.D. Shankishvili. Spherical hypersingular operators of imaginary order and their multipliers. *Fract. Calc. Appl. Anal.*, 4(1):101–112, 2001.

[33] B.G. Vakulov, N.K. Karapetiants, and L.D. Shankishvili. Spherical potentials of complex order in generalized Hölder spaces (in Russian). *Izv. Nats. Akad. Nauk Armenii Mat.*, 36(2):54–78 (2002), 2001. English transl. in J. Contemp. Math. Anal. 36 (2001), no. 2, 5–29 (2002).

[34] B.G. Vakulov, N.K. Karapetiants, and L.D. Shankishvili. Spherical potentials of complex order in generalized weighted Hölder spaces (in Russian). *Dokl. Akad. Nauk*, 382(3):301–304, 2002.

[35] B.G. Vakulov, N.K. Karapetiants, and L.D. Shankishvili. Spherical convolution operators with a power-logarithmic kernel in generalized Hölder spaces (Russian). *Izv. Vyssh. Uchebn. Zaved. Mat.*, (2):3–14, 2003. English transl. in Russian Math. (Iz. VUZ) 47 (2003), no. 2, 1–12.