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Spaces $\mathrm{BMO}^{p(\cdot)}(\mathbb{D})$ of variable order $p(z)$

by

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Abstract

We introduce and describe the spaces $\mathrm{BMO}^{p(\cdot)}(\mathbb{D})$, $1 \leq p(z) < \infty$ of functions of bounded mean oscillation over unit disc in the complex plane in the hyperbolic Bergman metric with respect to the Lebesgue measure and variable order of integrability $p = p(z)$.

Keywords: Bounded mean oscillation, Berezin transform, variable exponent
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1 Introduction

In this paper we introduce and describe the variable order space $\mathrm{BMO}^{p(\cdot)}(\mathbb{D})$, $1 \leq p(z) < \infty$ of functions of bounded mean oscillation in the hyperbolic Bergman metric with respect to the Lebesgue measure $d\mu(z) = \frac{1}{\pi} dxdy$, $z = x + iy$ on the unit disc $\mathbb{D}$ in complex plane $\mathbb{C}$. In this context the Berezin transform of the function φ plays a role of an average. The motivation of such a definition is that the Berezin transform plays a role of Poisson transform in complex analysis, and is covariant with respect to the analytic operations. Recently the Berezin transform was used in different contexts, starting with Hardy spaces ([12]) and including Bargman - Segal spaces ([2]), and, it was essentially used in the definition of (analytic) Bloch space and spaces of bounded mean oscillation ([15]).

The spaces $\mathrm{BMO}^{p}(\mathbb{D})$ of functions of bounded mean oscillation in the hyperbolic Bergman metric were defined and studied in [15], and this research continued in [2]. In the mentioned papers the spaces $\mathrm{BMO}^{p}(\mathbb{D})$ were defined in terms of averages over hyperbolic Bergman discs (see (4.2) below), but the authors provided an equivalent description in terms related to the Berezin transform. These results were obtained in the case $p = 2$; further some particular results were generalized for an arbitrary p, $1 \leq p < \infty$. These results are collected in [4] (see also [16], [13], [14] and references therein). The space $\mathrm{BMO}^{1}(\mathbb{D})$ was also studied in [17] in connection with the problem of relation between the compactness of a Toeplitz operator with symbol in $\mathrm{BMO}^{1}(\mathbb{D})$, acting in the Bergman space, and vanishing of its Berezin transform when approaching to the boundary. Similar problem was studied in [5] in the context of weighted Bergman space, where some weighted analogues of the space $\mathrm{BMO}^{p}(\mathbb{D})$ were introduced and studied as well.

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A principal difference with the theory of real BMO space is the fact that the behaviour with respect to the radius of the corresponding (hyperbolic) discs is not of essence, it could be even fixed. This fact also motivates the alternative definition of such a space in terms of the Berezin transform. Another important difference is the dependence of such BMO space on the parameter p.

On the other hand, in the real analysis during last two decades there was a big rise of interest to the so called spaces with non-standard growth or generalized Lebesgue spaces $L^{p(\cdot)}$ with the exponent $p(\cdot)$ which may vary from point to point. We refer to the surveying papers [3], [6], [10] on harmonic analysis and operator theory in such spaces, see also references in these surveys. Many publications in this topic last years were inspired both by possible applications (shown in the book [8]) and also by purely mathematical interest. The latter was caused by the difficulties arising in the study: the standard means are no more applicable, these variable exponent spaces being not invariant with respect to translations or dilatations, as a result convolution operators with integrable kernels are no more bounded in general, Minkowsky integral inequality being a very rough mean, and so on. Meanwhile, a breakthrough in this field was made after the boundedness of the Hardy-Littlewood operator was proved in variable exponent spaces.

The variable exponent approach seems to be not yet obtained the corresponding development in the complex analysis setting.

In this paper we deal with the above-mentioned $\mathrm{BMO}^{p(\cdot)}(\mathbb{D})$-spaces with respect to variable exponent $p(z)$. The main results are given in Theorems 4.1 and 4.4.

2 Auxiliaries and definitions

Let $\mathbb{D}$ be the unit circle centered at the origin. The space $L^{p(\cdot)}(\mathbb{D})$, where $p : \mathbb{D} \to [1, \infty)$, consists of functions f measurable on $\mathbb{D}$ such that the integral (called also modular of f)

$$I_p(f) = \int_{\mathbb{D}} |f(z)|^{p(z)} d\mu(z), \qquad d\mu(z) = \frac{1}{\pi} dxdy, \tag{2.1}$$

if finite. Defined in this form, the space $L^{p(\cdot)}(\mathbb{D})$ is linear if and only if $\sup\limits_{z\in\mathbb{D}} p(z) < \infty$, which will be assumed in the sequel. The norm of f in the space $L^{p(\cdot)}(\mathbb{D})$ is defined as

$$\|f\|_{L^{p(\cdot)}(\mathbb{D})} = \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\} \tag{2.2}$$

Set $p_* = \min\limits_{z\in\mathbb{D}} p(z)$, $p^* = \max\limits_{z\in\mathbb{D}} p(z)$. The Hölder inequality for functions of variable order of integrability is known to hold in the form

$$\left|\int_{\mathbb{D}} f(z)g(z)d\mu(z)\right| \le \kappa \, \|f\|_{L^{p(\cdot)}(\mathbb{D})}\|g\|_{L^{q(\cdot)}(\mathbb{D})}, \quad \frac{1}{p(z)} + \frac{1}{q(z)} = 1, \tag{2.3}$$

where $k = \frac{1}{p_*} + \frac{1}{q_*}$. We refer to [11], [7], [9] for further details on variable exponent spaces $L^{p(\cdot)}(\mathbb{D})$.

Let Ω - be a simply connected domain in $\mathbb{C}$ with smooth boundary. The Bergman metric $\beta_\Omega(z,w)$ for Ω is defined as

$$ds^2 = \frac{1}{2}\frac{\partial^2 \ln K_\Omega(z,z)}{\partial z \partial \overline{z}} d\mu_\Omega(z),$$

where $K_\Omega(z,w)$ - is the Bergman kernel for Ω (see, for instance, [4], [13], [14]), and $d\mu_\Omega(z)$ - is the Euclidean area element. The Bergman metric length element dl_{β_Ω} is connected to the Euclidean one dl by the rule $dl_{\beta_\Omega} = \sqrt{ds^2}dl$. In particular, for the unit disc one has $ds^2 = \frac{1}{(1-|z|^2)^2} d\mu(z)$, and the Bergman distance is defined by the formula

$$\beta(z,w) = \frac{1}{2}\ln\frac{1+|\alpha_z(w)|}{1-|\alpha_z(w)|} = \frac{1}{2}\ln\frac{|1-z\overline{w}|+|z-w|}{|1-z\overline{w}|-|z-w|}, \quad z,w \in \mathbb{D}. \tag{2.4}$$

Here

$$w \to \alpha_z(w) = \frac{z-w}{1-\overline{z}w}, \ z \in \mathbb{D} \tag{2.5}$$

stands for the Moebius transform of the unit disc to itself. Recall, that the Moebius transform has the following properties: $\alpha_z^2(w) = w$ and the real Jacobian of $w \to \alpha_z(w)$ is

$$|\alpha_z'(w)|^2 = \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4}.$$

Therefore, one has

$$|k_z(\alpha_z(\omega))|^2 d\mu(\alpha_z(\omega)) = d\mu(\omega), \tag{2.6}$$

where the function

$$k_z(w) = \frac{1-|z|^2}{(1-\overline{z}w)^2}$$

stands for the normalized coherent state for the Bergman space $\mathcal{A}^2(\mathbb{D})$ on the unit disc.

The Berezin transform $\widetilde{\varphi}(z)$ of a locally integrable function φ, connected to the Bergman space $\mathcal{A}^2(\mathbb{D})$, has the form

$$\widetilde{\varphi}(z) = \int_{\mathbb{D}} \varphi(w)|k_z(w)|^2 d\mu(w).$$

The Berezin transform (introduced by F.A. Berezin in [1]) of a bounded in a Hilbert space operator T or a function is a complex valued real analytic function.

With the goal to generalize the spaces $\mathrm{BMO}^{\mathrm{p}}(\mathbb{D})$ mentioned in Introduction, we define the variable order space $\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$ of functions of bounded mean oscillation as a set of functions φ locally integrable on $\mathbb{D}$, for which the semi-norm

$$\|\varphi\|_{\#,\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})} = \sup_{z\in\mathbb{D}} \|\varphi\circ\alpha_z(\cdot) - \widetilde{\varphi}(z)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})}$$

is finite. This $\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$-space may be treated as a weighted space according to relation (2.7) below. Note that $\varphi \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$ yields

$$\int_{\mathbb{D}} |\varphi(w) - \widetilde{\varphi}(0)|^{p(w)} d\mu(w) < \infty,$$

where $\widetilde{\varphi}(0) = \int_{\mathbb{D}} \varphi(w) d\mu(w)$, so that $\varphi \in L^{p(\cdot)}(\mathbb{D})$ and hence $\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D}) \subset L^{p(\cdot)}(\mathbb{D})$. A norm in $\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$ can be introduced as follows

$$\|\varphi\|_{\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})} = \|\varphi\|_{\#,\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})} + |\widetilde{\varphi}(0)|.$$

Using (2.6) one has

$$\int_{\mathbb{D}} |\varphi \circ \alpha_z(w) - \widetilde{\varphi}(z)|^{p \circ \alpha_z(w)} \, d\mu(w) = \int_{\mathbb{D}} |\varphi(w) - \widetilde{\varphi}(z)|^{p(w)} \, |k_z(w)|^2 \, d\mu(w). \quad (2.7)$$

and, correspondingly,

$$\|\varphi\|_{\#,\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})} = \sup_{z \in \mathbb{D}} \| \, (\varphi(\cdot) - \widetilde{\varphi}(z)) |k_z(\cdot)|^{\frac{2}{p(\cdot)}} \, \|_{L^{p(\cdot)}(\mathbb{D})}$$

3 Some results for functions $f \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$.

In the sequel κ stands for the constant in the Hölder inequality (2.3).

Lemma 3.1 *Let a function ψ be defined on $\mathbb{D}$. Then for $\varphi \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$ one has*

$$\|\varphi \circ \alpha_z(\cdot) - \widetilde{\varphi}(z)\|_{L^{p \circ \alpha_z(\cdot)}(\mathbb{D})} \leqslant (1 + \kappa) \|\varphi \circ \alpha_z(\cdot) - \psi(z)\|_{L^{p \circ \alpha_z(\cdot)}(\mathbb{D})}.$$

Proof. It is easy to check that

$$\begin{aligned} |\varphi(w) - \widetilde{\varphi}(z)| \;&\leqslant\; |\varphi(w) - \psi(z)| + |\psi(z) - \widetilde{\varphi}(z)| = |\varphi(w) - \psi(z)| + \\ &+\; \left| \int_{\mathbb{D}} (\varphi(u) - \psi(z)) |k_z(u)|^2 d\mu(u) \right|. \end{aligned}$$

Hence,

$$\begin{aligned} \|\varphi \circ \alpha_z(\cdot) - \widetilde{\varphi}(z)\|_{L^{p \circ \alpha_z(\cdot)}(\mathbb{D})} \;&\leqslant\; \|\varphi \circ \alpha_z(\cdot) - \psi(z)\|_{L^{p \circ \alpha_z(\cdot)}(\mathbb{D})} + \\ &+\; \left| \int_{\mathbb{D}} (\varphi(u) - \psi(z)) |k_z(u)|^2 d\mu(u) \right|. \end{aligned}$$

Here we use the fact that $|k_z(w)|^2 d\mu(w)$ is a probability measure on $\mathbb{D}$. Now it is sufficient to note, that

$$\left| \int_{\mathbb{D}} (\varphi(u) - \psi(z)) |k_z(u)|^2 d\mu(u) \right| \leqslant \kappa \|\varphi \circ \alpha_z(\cdot) - \psi(z)\|_{L^{p \circ \alpha_z(\cdot)}(\mathbb{D})},$$

where we used (2.7) and the Holder inequality. □

Corollary 3.2 *The space* $\mathrm{BMO}^{p(\cdot)}(\mathbb{D})$ *can be equivalently described as a space of functions* φ *for which*

$$\sup_{z\in\mathbb{D}}\inf_{\delta\in\mathbb{C}}\int_{\mathbb{D}}|\varphi(w)-\delta|^{p(w)}|k_z(w)|^2d\mu(w)<\infty,$$

and moreover,

$$\|\varphi\|'_{\#,\mathrm{BMO}^{p(\cdot)}(\mathbb{D})}\leqslant\|\varphi\|_{\#,\mathrm{BMO}^{p(\cdot)}(\mathbb{D})}\leqslant(1+\kappa)\|\varphi\|'_{\#,\mathrm{BMO}^{p(\cdot)}(\mathbb{D})},\tag{3.1}$$

where

$$\|\varphi\|'_{\#,\mathrm{BMO}^{p(\cdot)}(\mathbb{D})}=\sup_{z\in\mathbb{D}}\inf_{\delta\in\mathbb{C}}\|\ (\varphi(\cdot)-\delta)|k_z(\cdot)|^{\frac{2}{p(\cdot)}}\ \|_{L^{p(\cdot)}(\mathbb{D})}.\tag{3.2}$$

Proof. Indeed, for a fixed $z\in\mathbb{D}$ one has

$$\inf_{\delta\in\mathbb{C}}\int_{\mathbb{D}}|\varphi(w)-\delta|^{p(w)}|k_z(w)|^2d\mu(w)\leqslant\int_{\mathbb{D}}|\varphi(w)-\widetilde{\varphi}(z)|^{p(w)}|k_z(w)|^2d\mu(w),$$

which implies

$$\|\varphi\|'_{\#,\mathrm{BMO}^{p(\cdot)}(\mathbb{D})}\leqslant\|\varphi\|_{\#,\mathrm{BMO}^{p(\cdot)}(\mathbb{D})}.$$

The second inequality in (3.1) follows from Lemma 3.1, taking $\psi(z)=\delta$. □

Corollary 3.3 *If* $\varphi\in\mathrm{BMO}^{p(\cdot)}(\mathbb{D})$, *then* $|\varphi|\in\mathrm{BMO}^{p(\cdot)}(\mathbb{D})$. *The inverse statement is not true, in general.*

Proof. Due to Lemma 3.1, instead of proving the boundedness of

$$\|\ |\varphi|\circ\alpha_z(\cdot)-\widetilde{|\varphi|}(z)\ \|_{L^{p(\cdot)}(\mathbb{D})},$$

it is sufficient to prove the boundedness of

$$\|\ |\varphi|\circ\alpha_z(\cdot)-|\widetilde{\varphi}(z)|\ \|_{L^{p(\cdot)}(\mathbb{D})}.$$

We have

$$\int_{\mathbb{D}}|\ |\varphi\circ\alpha_z(w)|-|\widetilde{\varphi}(z)|\ |^{p(w)}\ d\mu(w)\leqslant\int_{\mathbb{D}}|\ \varphi\circ\alpha_z(w)-\widetilde{\varphi}(z)\ |^{p(w)}\ d\mu(w),$$

which according to the definition of norm in $L^{p(\cdot)}(\mathbb{D})$ completes the proof. □

Lemma 3.4 *The inclusion* $\mathrm{BMO}^{p(\cdot)}(\mathbb{D})\subset\mathrm{BMO}^1(\mathbb{D})$ *holds.*

Proof. According to Holder inequality (2.3):

$$\begin{aligned}&\|\ (\varphi(\cdot)-\widetilde{\varphi}(z))|k_z(\cdot)|^2\ \|_{L^1(\mathbb{D})}\leqslant\kappa\|\ (\varphi(\cdot)-\widetilde{\varphi}(z))|k_z(\cdot)|^{\frac{2}{p(\cdot)}}\ \|_{L^{p(\cdot)}(\mathbb{D})}\times\\&\times\|\ |k_z(\cdot)|^{\frac{2}{q(\cdot)}}\ \|_{L^{q(\cdot)}(\mathbb{D})}=\kappa\|\ (\varphi(\cdot)-\widetilde{\varphi}(z))|k_z(\cdot)|^{\frac{2}{p(\cdot)}}\ \|_{L^{p(\cdot)}(\mathbb{D})}\end{aligned}$$

where, as usual, $\frac{1}{p(z)}+\frac{1}{q(z)}=1$, and we noticed again that $|k_z(w)|^2d\mu(w)$ is a probability measure on $\mathbb{D}$. Taking supremum over $z\in\mathbb{D}$ finishes the proof. □

Since $\mathrm{BMO}^{p(\cdot)}(\mathbb{D})\subset\mathrm{BMO}^1(\mathbb{D})$, then due to results from [4] (pages 46-48) we have the following statement.

Lemma 3.5 *If $\varphi \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$, then the function $\widetilde{\varphi}$ satisfies the Lipschitz condition in the Bergman metric:*

$$|\widetilde{\varphi}(z) - \widetilde{\varphi}(w)| \leqslant C\beta(z, w).$$

Even though the definition of the spaces of functions of bounded mean oscillation in [4] differs from that we use here (in [4] it is given in terms of averages, see also section 5 below for the case of variable exponent $p(z)$), the equivalence of these definitions for the case of constant p is known (see also [5]).

4 A characterization of functions from $\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$

In the following theorem we give certain sufficient and certain necessary conditions for a function to be in $\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$.

Theorem 4.1 *The following condition*

$$\sup_{z\in\mathbb{D}}\{\ \|\varphi\circ\alpha_z(\cdot)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})} - |\widetilde{\varphi}(z)|\ \} < \infty \tag{4.1}$$

is necessary and, if $\widetilde{\varphi}$ is bounded, it is also sufficient for a function φ to be in $\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$.

Proof. Write

$$\|\varphi\circ\alpha_z(\cdot)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})} \leqslant \|\varphi\circ\alpha_z(\cdot) - \widetilde{\varphi}(z)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})} + |\widetilde{\varphi}(z)|.$$

Note that

$$\begin{aligned} |\widetilde{\varphi}(z)| &= \left|\int_{\mathbb{D}} \varphi(w)\ |k_z(w)|^2 d\mu(w)\right| \leqslant \int_{\mathbb{D}} |\varphi(w)|\ |k_z(w)|^2 d\mu(w) = \\ &= \int_{\mathbb{D}} |\varphi\circ\alpha_z(w)|\ d\mu(w) \leqslant \kappa\|\varphi\circ\alpha_z(\cdot)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})}. \end{aligned}$$

Hence, for $\varphi \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$ one has

$$\sup_{z\in\mathbb{D}}\{\ \|\varphi\circ\alpha_z(\cdot)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})} - |\widetilde{\varphi}_\lambda(z)|\ \} \leqslant C\|\varphi\|_{\#,\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})} < \infty,$$

which proves the necessity of condition (4.1).

To prove that (4.1) is sufficient in case the function $\widetilde{\varphi}$ is bounded, consider the inequality

$$\|\varphi\circ\alpha_z(\cdot) - \widetilde{\varphi}(z)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})} \ \leqslant\ \|\varphi\circ\alpha_z(\cdot)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})} + |\widetilde{\varphi}(z)|.$$

If the function $\widetilde{\varphi}$ is bounded and inequality (4.1) is satisfied, then $\|\varphi\circ\alpha_z(\cdot)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})}$ is also bounded and consequently $\|\varphi\|_{\#,\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})} < \infty$. □

For a function φ locally integrable on $\mathbb{D}$ and $0 < r < \infty$ introduce the hyperbolic Bergman average

$$\widehat{\varphi}_r(z) = \frac{1}{|\mathrm{D}(z,r)|}\int_{\mathrm{D}(z,r)} \varphi(w)d\mu(w), \tag{4.2}$$

where

$$\mathrm{D}(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\} \subset \mathbb{D}$$

is a disc in the Bergman metric with (hyperbolic) center in z and (hyperbolic) radius r. It is known ([4]) that

$$C^{-1} < |k_z(w)|^2|\mathrm{D}(z,r)| < C,\ w \in \mathrm{D}(z,r), \tag{4.3}$$

where $|\mathrm{D}(z,r)|$ is the Euclidean area of $\mathrm{D}(z,r)$ with respect to the Lebesgue measure $d\mu$:

$$|\mathrm{D}(z,r)| = \int_{\mathrm{D}(z,r)} d\mu(w) = \left(\frac{(1-|z|^2)\tanh r}{1-|z|^2\tanh^2 .r}\right)^2$$

Lemma 4.2 *Let φ be a function locally integrable on $\mathbb{D}$ and $0 < r < \infty$. Then the following pointwise inequality holds:*

$$|\widehat{\varphi}_r(z) - \widetilde{\varphi}(z)| \leqslant C\|\varphi \circ \alpha_z(\cdot) - \widetilde{\varphi}(z)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})},\quad z \in \mathbb{D}. \tag{4.4}$$

Proof. Making use of inequality (4.3) one has

$$\begin{aligned}
|\widehat{\varphi}_r(z) - \widetilde{\varphi}(z)| &\leqslant \frac{1}{|\mathrm{D}(z,r)|}\int_{\mathrm{D}(z,r)} |\varphi(w) - \widetilde{\varphi}(z)|d\mu(w) \leqslant \\
&\leqslant C\int_{\mathrm{D}(z,r)} |\varphi(w) - \widetilde{\varphi}(z)||k_z(w)|^2 d\mu(w) \leqslant \\
&\leqslant C\int_{\mathbb{D}} |\varphi(w) - \widetilde{\varphi}(z)||k_z(w)|^2 d\mu(w) \leqslant \\
&\leqslant C\|\varphi \circ \alpha_z(\cdot) - \widetilde{\varphi}(z)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})}.
\end{aligned}$$

□

Corollary 4.3 *If $\varphi \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$, then the function $\widetilde{\varphi}$ is bounded if and only if the function $\widehat{\varphi}_r$ is bounded for a fixed $r > 0$.*

Gathering the results obtained, we arrive at the following final statement.

Theorem 4.4 *For a function φ locally integrable on $\mathbb{D}$, the following conditions are equivalent:*

1. *the function $\widetilde{\varphi}$ is bounded and $\varphi \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$;*
2. *the function $\widehat{\varphi}_r$ is bounded for an $r > 0$ and $\varphi \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$;*

3. *the function* $\widetilde{\varphi}$ *is bounded and*

$$\sup_{z\in\mathbb{D}}\{\ \|\varphi\circ\alpha_z(\cdot)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})}-|\widetilde{\varphi}(z)|\ \}<\infty;$$

4. *the function* $\widehat{\varphi}_r$ *is bounded an* $r>0$ *and*

$$\sup_{z\in\mathbb{D}}\{\ \|\varphi\circ\alpha_z(\cdot)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})}-|\widetilde{\varphi}(z)|\ \}<\infty;$$

5. *the function* $\|\varphi\circ\alpha_z(\cdot)\|_{L^{p\circ\alpha_z(\cdot)}(\mathbb{D})}$ *is bounded.*

5 Description in terms of means

Given $r>0$, we say that f satisfies $(r,p(\cdot))$ - condition if

$$\sup_{z\in\mathbb{D}}\|(\varphi-\widehat{\varphi}_r(z))|\mathrm{D}(z,r))|^{-\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\mathrm{D}(z,r))}<\infty. \tag{5.5}$$

The condition (5.5) is equivalent to the following one

$$\sup_{z\in\mathbb{D}}\frac{1}{\mathrm{D}(z,r))}\int_{\mathrm{D}(z,r))}|\varphi(w)-\widehat{\varphi}_r(z)|^{p(w)}\,d\mu(w)<\infty. \tag{5.6}$$

Indeed, set

$$F_{z,r}(w)=(\varphi(w)-\widehat{\varphi}_r(z))|\mathrm{D}(z,r))|^{-\frac{1}{p(w)}}\chi_{\mathrm{D}(z,r))},$$

where $\chi_{\mathrm{D}(z,r))}$ stands for the characteristic function of the hyperbolic disc $\mathrm{D}(z,r))$. Then condition (5.5) is

$$\sup_{z\in\mathbb{D}}\|F_{z,r}\|_{L^{p(\cdot)}(\mathbb{D})}<\infty,$$

and condition (5.6) is nothing but

$$\sup_{z\in\mathbb{D}}I_p(F_{z,r})<\infty,$$

where the modular $I_p(f)$ was defined in (2.1). As is well known, $\|f\|^{p^*}_{L^{p(\cdot)}(\mathbb{D})}\leqslant I_p(f)\leqslant\|f\|^{p^*}_{L^{p(\cdot)}(\mathbb{D})}$ if $\|f\|_{L^{p(\cdot)}(\mathbb{D})}\leqslant 1$, and $\|f\|^{p_*}_{L^{p(\cdot)}(\mathbb{D})}\leqslant I_p(f)\leqslant\|f\|^{p_*}_{L^{p(\cdot)}(\mathbb{D})}$ if $\|f\|_{L^{p(\cdot)}(\mathbb{D})}\geqslant 1$. Hence,

$$c_1\leqslant\|f\|_{L^{p(\cdot)}(\mathbb{D})}\leqslant c_2\quad\text{implies}\quad c_3\leqslant I_p(f)\leqslant c_4,$$

and

$$C_1\leqslant I_p(f)\leqslant C_2\quad\text{implies}\quad C_3\leqslant\|f\|_{L^{p(\cdot)}(\mathbb{D})}\leqslant C_4,$$

where $c_3=\min\{c_1^{p_*},c_1^{p^*}\}$, $c_4=\min\{c_2^{p_*},c_2^{p^*}\}, C_3=\min\{C_1^{1/p_*},C_1^{1/p^*}\}$, $C_4=\min\{C_2^{1/p_*},C_2^{1/p^*}\}$. Hence, if the set of norms $\|F_{z,r}\|_{L^{p(\cdot)}(\mathbb{D})}$ parameterized by $z\in\mathbb{D}$ has lower and upper bounds c_1 and c_2, then the corresponding set $I_p(F_{z,r})$,

$z \in \mathbb{D}$ has bounds c_3 and c_4 which depend only on c_1, c_2 and p_*, p^*. Thus, (5.5) implies (5.6). The inverse implication is analogous.

Note that condition (5.6) can be rewritten as $\sup_{z\in\mathbb{D}} \left(|\varphi - \widehat{\varphi}_r|^{p(\cdot)}\right)_r^{\widehat{}}(z) < \infty$. Also, in the case of constant $p(z) = p$ for a locally integrable function f satisfying to any fixed (r, p) condition is equivalent to be in $\mathrm{BMO}^p(\mathbb{D})$. The situation is more complicated for variable order of integration - we managed to show that this condition is sufficient for a function to be in $\mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$.

Below we list preliminary results - some of them are automatically extendable from the case of constant order of integration [5], but the proofs of the others differ from that case due to specifics of variable order of integration.

Lemma 5.1 *Let $\nu \in \mathbb{R}$, and $\tau = \tau(z) > 0$ be bounded on $\mathbb{D}$. Then*

$$J_{\nu,\tau} = \int_{\mathbb{D}} \left(\int_{\mathbb{D}} (1 + \beta(u, v))^{\nu} d\mu(v) \right)^{\tau(u)} d\mu(u) < \infty. \tag{5.7}$$

Proof. The case $\tau(u) = 1$, $\nu = 2$ was treated in [4]. Assume $\nu > 0$ (the case $\nu \leqslant 0$ is obvious). Set $v = \alpha_u(w)$ and having in mind (2.6) calculate

$$\begin{aligned}
J_{\nu,\tau} &= \int_{\mathbb{D}} d\mu(u) \left(\int_{\mathbb{D}} \left(1 + \frac{1}{2} \ln \frac{1 + |\alpha_u(v)|}{1 - |\alpha_u(v)|} \right)^{\nu} d\mu(v) \right)^{\tau(u)} = \\
&= \int_{\mathbb{D}} d\mu(u) \left(\int_{\mathbb{D}} \left(1 + \frac{1}{2} \ln \frac{1 + |w|}{1 - |w|} \right)^{\nu} |k_u(w)|^2 d\mu(w) \right)^{\tau(u)} \leqslant \\
&\leqslant C \int_{\mathbb{D}} d\mu(u) \left(\int_{\mathbb{D}} (1 - |w|)^{-\nu\varepsilon} |k_u(w)|^2 d\mu(w) \right)^{\tau(u)} = \\
&= C \int_{\mathbb{D}} (1 - |u|^2)^{2\tau(u)} d\mu(u) \left(\int_{\mathbb{D}} \frac{(1 - |w|)^{-\nu\varepsilon}}{|1 - \overline{u}w|^4} d\mu(w) \right)^{\tau(u)}.
\end{aligned}$$

Choose ε such that $\varepsilon\nu\tau(u) < 1$ and use (see [4], Theorem 1.7):

$$\int_{\mathbb{D}} \frac{(1 - |w|)^{-\nu\varepsilon}}{|1 - \overline{u}w|^4} d\mu(w) \leqslant C_1 (1 - |u|^2)^{-2-\varepsilon\nu}, \quad u \in \mathbb{D}.$$

We have

$$J_{\nu,\tau} \leqslant C_2 \int_{\mathbb{D}} (1 - |u|^2)^{-\varepsilon\nu\tau(u)} d\mu(u) < \infty.$$

□

Proposition 5.2 *Let φ be nonnegative and $p(z)$ - locally integrable over $\mathbb{D}$. Then the following statements are equivalent*

1. $\sup_{z\in\mathbb{D}} \widehat{\varphi^{p(\cdot)}}_s(z) < \infty, \quad 0 < s < \infty,$
2. $\sup_{z\in\mathbb{D}} \widetilde{\varphi^{p(\cdot)}}(z) < \infty.$

PROOF. The proof follows immediately from the case of constant $p \in [1, \infty)$ (see [5]) by replacing φ with $\varphi^{p(\cdot)}$. $\square$

Proposition 5.3 *Let φ satisfies $(r, p(\cdot))$ condition for fixed $0 < r < \infty$. Then for any $z, w \in \mathbb{D}$ and arbitrary fixed s we have*

$$|\widehat{\varphi}_s(z) - \widehat{\varphi}_s(w)| \leqslant C[\beta(z, w) + 1] \tag{5.8}$$

PROOF. Let φ satisfy the $(r, p(\cdot))$ - condition in the form (5.6), $E_z(\varphi) = \{z \in \mathbb{D} : |\varphi(w) - \widehat{\varphi}(z)| \geqslant 1\}$ and $G_z(\varphi) = \mathbb{D} \setminus E_z(\varphi)$ is its complement in $\mathbb{D}$. Consider

$$\begin{aligned}
\sup_{z\in\mathbb{D}} \widehat{(|\varphi - \widehat{\varphi}_s|^{p_*})}_s(z) &= \sup_{z\in\mathbb{D}} \frac{1}{\mathrm{D}(z,r))} \int_{\mathrm{D}(z,r))} |\varphi(w) - \widehat{\varphi}_r(z)|^{p_*} \, d\mu(w) \leqslant \\
&\leqslant \sup_{z\in\mathbb{D}} \frac{1}{\mathrm{D}(z,r))} \int_{\mathrm{D}(z,r))\cap E_z(\varphi)} |\varphi(w)\widehat{\varphi}_r(z)|^{p(w)} \, d\mu(w) + \\
&+ \sup_{z\in\mathbb{D}} \frac{1}{\mathrm{D}(z,r))} \int_{\mathrm{D}(z,r))\cap G_z(\varphi)} d\mu(w).
\end{aligned}$$

The first term is bounded by the left-hand side of (5.6) and the second one is less than 1. Hence, φ satisfies the (r, p_*) condition, and again the result is due to the case of constant $p(z) = p$ ([5]). $\square$

Lemma 5.4 *Let φ satisfy the $(r, p(\cdot))$ condition for fixed $r \in (0, \infty)$. Then $\widehat{\varphi}_s \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$ for every $s \in (0, \infty)$.*

PROOF. Using (5.8) and changing the variables according to (2.6) we have

$$\begin{aligned}
&\left|\widehat{\varphi}_s \circ \alpha_z(w) - \widetilde{\widehat{\varphi}_s}(z)\right| = \left|\widehat{\varphi}_s \circ \alpha_z(w) - \int_{\mathbb{D}} \widehat{\varphi}_s(u)|k_z(u)|^2 d\mu(u)\right| \leqslant \\
&\leqslant \int_{\mathbb{D}} |\widehat{\varphi}_s \circ \alpha_z(w) - \widehat{\varphi}_s(u)| \, |k_z(u)|^2 d\mu(u) \leqslant C \int_{\mathbb{D}} (1 + \beta(u, \alpha_z(w))) \, |k_z(u)|^2 d\mu(u) = \\
&= C \int_{\mathbb{D}} (1 + \beta(v, w)) d\mu(v).
\end{aligned}$$

Now the result follows by (5.7). $\square$

Lemma 5.5 *Let φ satisfy the $(r, p(\cdot))$-condition for fixed $r \in (0, \infty)$. Then $\varphi - \widehat{\varphi}_s \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$ for every $0 < s < \infty$.*

PROOF. Consider

$$\begin{aligned}
\|(\varphi - \widehat{\varphi}_s)|\mathrm{D}(z,s)|^{-\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\mathrm{D}(z,s))} &\leqslant \|(\varphi - \widehat{\varphi}_s(z))|\mathrm{D}(z,s)|^{-\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\mathrm{D}(z,s))} + \\
&+ \|(\widehat{\varphi}_s - \widehat{\varphi}_s(z))|\mathrm{D}(z,s)|^{-\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\mathrm{D}(z,s))}
\end{aligned}$$

The first term is bounded uniformly in z according to (5.5). To estimate the second one recall that $w \in \mathrm{D}(z,s)$ means that $\beta(z,w) < s$, which along with (5.8) gives

$$\begin{aligned} \|(\widehat{\varphi}_s - \widehat{\varphi}_s(z))|\mathrm{D}(z,s)|^{-\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\mathrm{D}(z,s))} &\leqslant \|C(s+1)|\mathrm{D}(z,s)|^{-\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\mathrm{D}(z,s))} = \\ &= C(s+1) \end{aligned}$$

Hence, by the same arguments as in the beginning of this section, one can show that $\widehat{\left(|\varphi - \widehat{\varphi}_s|^{p(\cdot)}\right)}_s(z)$ is bounded while z runs over $\mathbb{D}$. Due to Proposition 5.2, the function $\widetilde{\left(|\varphi - \widehat{\varphi}_s|_s^{p(\cdot)}\right)}(z)$ is bounded. But

$$\widetilde{\left(|\varphi - \widehat{\varphi}_s|_s^{p(\cdot)}\right)}(z) = \int_{\mathbb{D}} |(\varphi - \widehat{\varphi}_s)(w)|^{p(w)} |k_z(w)|^2 d\mu(w).$$

The expression above is the $p(\cdot)$ - modular of the function

$$F_{z,s}(w) = (\varphi - \widehat{\varphi}_s)(w)|k_z(w)|^{\frac{2}{p(w)}})$$

Again, by the mentioned arguments, the $p(\cdot)$ norms of $F_{z,s}$ are bounded for $z \in \mathbb{D}$, but according the the change of variable rule (2.6) this is nothing else but condition 5 in Theorem 4.4, which gives $\varphi - \widehat{\varphi}_s \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$. □

Now we are in position to formulate the final result.

Theorem 5.6 *Let φ satisfy the $(r, p(\cdot))$-condition for fixed $r \in (0,\infty)$. Then $\varphi \in \mathrm{BMO}^{\mathrm{p}(\cdot)}(\mathbb{D})$.*

Proof. Write $\varphi = (\varphi - \widehat{\varphi}_s) + \widehat{\varphi}_s$ and use previous lemmas. □

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