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8. Weighted Boundedness of the Maximal, Singular and Potential Operators in Variable Exponent Spaces

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We present a brief survey of recent results on boundedness of some classical operators within the frameworks of weighted spaces $L^{p(\cdot)}(\varrho)$ with variable exponent $p(x)$, mainly in the Euclidean setting and dwell on a new result of the boundedness of the Hardy-Littlewood maximal operator in the space $L^{p(\cdot)}(X,\varrho)$ over a metric measure space X satisfying the doubling condition. In the case where X is bounded, the weight function satisfies a certain version of a general Muckenhoupt-type condition For a bounded or unbounded X we also consider a class of weights of the form $\varrho(x) = [1+d(x_0,x)]^{\beta_\infty}\prod_{k=1}^m w_k(d(x,x_k))$, $x_k \in X$, where the functions $w_k(r)$ have finite upper and lower indices $m(w_k)$ and $M(w_k)$.
Some of the results are new even in the case of constant p.

KEY WORDS: maximal functions, potential operators, weighted Lebesgue spaces, weighted estimates, variable exponent, metric measure space, doubling condition, Zygmund conditions, Zygmund-Bary-Stechkin class

MSC (2000): 42B25, 47B38

1. INTRODUCTION

In this paper, based on the lecture *Harmonic Analysis on variable exponent spaces* presented by the second author at AMADE International Conference AMADE-2006 (Analytic Methods of Analysis and Differential Equations) held in "Staiki", Minsk, Belarus, September 13-19, 2006, we present a survey of a certain selection of known results on weighted estimations of operators in variable exponent Lebesgue spaces and expose some new results.

The study of classical operators of harmonic analysis (maximal, singular operators and potential type operators) in the generalized Lebesgue spaces $L^{p(\cdot)}$ with variable exponent, weighted or non-weighted, undertaken last decade, continues to attract a strong interest of researchers, influenced in particular by possible applications revealed in the book [88]. We refer in particular to the surveying articles [21], [63], [99].

The area which is now called variable exponent analysis, last decade became a rather branched field with many interesting results obtained in Harmonic Analysis, Approximation Theory, Operator Theory, Pseudo-Differential Operators, the survey of which would be a good task. Our survey is far from being complete in this sense. In our selection of papers for this survey we have mainly chosen results on weighted estimates, obtained after the above surveys had appeared, and present some new results on weighted boundedness of the Hardy-Littlewood maximal operator in such spaces over metric measure spaces.

The paper is organized as follows. In Section 2. we recall some basics from the theory of generalized Lebesgue spaces with variable exponent on metric measure spaces. In Section 3. we give necessary preliminaries on the upper and lower indices (of Matuszewska-Orlicz-type) of weights in the Zygmund-Bary-Stechkin class. In Section 4. we give a survey of recent results on the weighted boundedness of classical operators of harmonic analysis (Hardy-Littlewood maximal operator, Calderón-Zygmund type singular operators, Cauchy singular integral operator on Carleson curves, potential type operators and some classes of convolution operators) in weighted Lebesgue spaces with variable exponent.

In Section 5. we give new results - Theorems A, B and C - on the weighted boundedness of the maximal operator on metric measure spaces with doubling condition.

Theorem A gives a kind of ”Muckenhoupt-looking” condition. In Theorem B we deal with a bounded space X and radial-type oscillating weights $w[d(x_0, x)]$; we obtain sufficient conditions for the boundedness in this case in the form

$$-\frac{m(\mu B)}{p(x_0)} < m(w) \leq M(w) < \frac{m(\mu B)}{p'(x_0)}, \tag{1.1}$$

where

$$m(\mu B) = \lim_{t\to 0} \frac{\ln\left(\limsup\limits_{r\to 0}\inf\limits_{x\in X} \frac{\mu B(x,rt)}{\mu B(x,r)}\right)}{\ln t} \tag{1.2}$$

is a kind of ”uniform” lower Matuszewska-Orlicz-type index of the function $\mu B(x, r)$, $B(x, r) = \{y \in X : d(x, y) < r\}$ with respect to the variable r: see Section 3. for Matuszewska-Orlicz-type indices where one may find hints onto how these indices of the measure $\mu B(x, r)$ appeared. In Theorem C we give a version of Theorem B for the case of unbounded spaces X.

Notation

(X, d, μ) is a measure space with quasimetric d and a non-negative measure μ;
$B(x,r) = B_X(x,r) = \{y \in X : d(x,y) < r\}$;
$p'(x) = \frac{p(x)}{p(x)-1}$, $1 < p(x) < \infty$, $\frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1$;
$p_- = p_-(X) = \inf\limits_{x\in X} p(x)$, $p^+ = p^+(X) = \sup\limits_{x\in X} p(x)$;
$p'_- = \inf\limits_{x\in X} p'(x) = \frac{p^+}{p^+-1}$, $(p')^+ = \sup\limits_{x\in X} p'(x) = \frac{p_-}{p_--1}$;
$\mathbb{P}(X)$, see (2.2)-(2.3);
C, c may denote different positive constants;
a.i. =almost increasing $\Longleftrightarrow u(x) \le Cu(y)$ for $x \le y, C > 0$.

2. SOME BASICS FOR VARIABLE EXPONENT SPACES

In the sequel (X, d, μ) is a homogeneous type space, i.e. a measure space with quasimetric d and a non-negative measure μ satisfying the doubling condition; we refer to [42], [45], [53] for the basic notions of function spaces on metric measure spaces. We suppose that the measure μ satisfies the doubling condition

$$\mu B(x, 2r) \le C\mu B(x,r). \tag{2.1}$$

By $\mathbb{P}(X)$ we denote the set of bounded measurable functions $p(x)$ defined on X which satisfy the conditions

$$1 < p_- \le p(x) \le p^+ < \infty, \quad x \in X \tag{2.2}$$

and

$$|p(x) - p(y)| \le \frac{A}{\ln \frac{1}{d(x,y)}}\ , \quad d(x,y) \le \frac{1}{2}, \quad x,\, y \in X. \tag{2.3}$$

By $L^{p(\cdot)}(X, \varrho)$, where $\varrho(x) \ge 0$, we denote the weighted Banach space of measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}(X,\varrho)} := \|\varrho f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int\limits_X \left|\frac{\varrho(x)f(x)}{\lambda}\right|^{p(x)} d\mu(x) \le 1\right\} < \infty. \tag{2.4}$$

We write $L^{p(\cdot)}(X,1) = L^{p(\cdot)}(X)$ and $\|f\|_{L^{p(\cdot)}(X)} = \|f\|_{p(\cdot)}$ in the case $\varrho(t) \equiv 1$.

The generalized Lebesgue spaces $L^{p(\cdot)}(X)$ and Sobolev spaces $W^{1,p(\cdot)}$ with variable exponent on metric measure spaces have been considered in [26], [35], [40], [49], [50], [51], [61], [80], the Euclidean case being studied in [27], [33], [75], [102], see also references therein. We recall the Hölder inequality

$$\int\limits_X |f(x)g(x)|\, d\mu(x) \le k\big\|f\big\|_{p(\cdot)} \cdot \big\|g\big\|_{p'(\cdot)} \tag{2.5}$$

where $k = \frac{1}{p_-} + \frac{1}{p'_-}$. We note also that the embedding

$$L^{p(\cdot)} \subseteq L^{s(\cdot)}, \quad \|f\|_{s(\cdot)} \leq C\|f\|_{p(\cdot)}, \tag{2.6}$$

is valid for $1 \leq s(x) \leq p(x) \leq p^+ < \infty$, when $\mu(X) < \infty$.

3. PRELIMINARIES ON ZYGMUND-BARY-STECHKIN CLASSES

In this section we follow some ideas of papers [56], [90], [93], [94] . Let $0 < \ell \leq \infty$. We denote

$$W = \{w \in C([0,\ell]) : \ w(t) > 0 \ \text{ for } \ t > 0, \ \ w(t) \ \text{ is almost increasing}\} \tag{3.1}$$

and

$$W_0 = \{w \in W : w(0) = 0\}. \tag{3.2}$$

We also need a wider class

$$\widetilde{W}([0,\ell]) = \{\varphi : \ \exists a = a(\varphi) \in \mathbb{R}^1 \ \text{ such that } \ x^a\varphi(x) \in W([0,\ell])\}. \tag{3.3}$$

3.1 The Zygmund-Bary-Stechkin type classes $\Phi^\alpha_\beta = \Phi^\alpha_\beta([0,\ell])$ and $\Psi^\alpha_\beta = \Psi^\alpha_\beta([\ell,\infty])$, $0 < \ell < \infty$.

The following class Φ^α_β of Zygmund-Bary-Stechkin type in the case $\alpha = 0$ and $\beta = 1, 2, 3, ...$ was introduced in [4] (in [4] functions w were assumed to be increasing functions).

Definition 3.1. The Zygmund-Bary-Stechkin type class $\Phi^\alpha_\beta = \Phi^\alpha_\beta([0,\ell])$, $-\infty < \alpha < \beta < \infty$, is defined as $\Phi^\alpha_\beta := \mathcal{Z}^\alpha \cap \mathcal{Z}_\beta$, where $\mathcal{Z}^\alpha$ is the class of functions $w \in \widetilde{W}$ satisfying the condition

$$\int_0^h \frac{w(t)}{t^{1+\alpha}} dt \leq c\frac{w(h)}{h^\alpha} \tag{$\mathbb{Z}^\alpha$}$$

and $\mathcal{Z}_\beta$ is the class of functions $w \in W$ satisfying the condition

$$\int_h^\ell \frac{w(t)}{t^{1+\beta}} d(t) \leq c\frac{w(h)}{h^\beta}, \tag{$\mathbb{Z}_\beta$}$$

where $c = c(w) > 0$ does not depend on $h \in (0,\ell]$.

We also need a class of functions with a similar behaviour at infinity. Let $C_+([\ell,\infty))$, $0 < \ell < \infty$, be the class of functions $w(t)$ on $[\ell,\infty)$, continuous and positive at every point $t \in [\ell,\infty)$ and having a finite or infinite limit $\lim\limits_{t\to\infty} w(t) =: w(\infty)$. We denote

$$W = W([\ell,\infty)) = \{w \in C_+([\ell,\infty) : \ w(t) \ \text{ is a.i.}\} \tag{3.4}$$

and

$$\widetilde{W}([\ell,\infty)) = \{\varphi : \ \exists a = a(\varphi) \in \mathbb{R}^1 \ \text{ such that } \ x^a\varphi(x) \in W([\ell,\infty)\}. \tag{3.5}$$

Definition 3.2. Let $-\infty < \alpha < \beta < \infty$. We put $\Psi_\alpha^\beta := \widehat{\mathcal{Z}}^\beta \cap \widehat{\mathcal{Z}}_\alpha$, where $\widehat{\mathcal{Z}}^\beta$ is the class of functions $w \in \widetilde{W}([\ell,\infty))$ satisfying the condition

$$\int_r^\infty \left(\frac{r}{t}\right)^\beta \frac{w(t)\,dt}{t} \le cw(r), \qquad r \to \infty, \tag{3.6}$$

and $\widehat{\mathcal{Z}}_\alpha$ is the class of functions $w \in W([\ell,\infty))$ satisfying the condition

$$\int_\ell^r \left(\frac{r}{t}\right)^\alpha \frac{w(t)\,dt}{t} \le cw(r), \qquad r \to \infty \tag{3.7}$$

where $c = c(w) > 0$ does not depend on $r \in [\ell,\infty)$.

Observe that properties of functions in the class $\Psi_\alpha^\beta([\ell,\infty))$ are easily derived from those of functions in $\Phi_\beta^\alpha([0,\ell])$ because of the following equivalence

$$w \in \Psi_\alpha^\beta([\ell,\infty)) \quad \Longleftrightarrow \quad w_* \in \Phi_{-\alpha}^{-\beta}([0,\ell^*]), \tag{3.8}$$

where $w_*(t) = w\left(\frac{1}{t}\right)$ and $\ell_* = \frac{1}{\ell}$.

3.2 Index numbers $m(w)$ and $M(w)$ of non-negative a. i. functions

The numbers

$$m(w) = \sup_{t>1} \frac{\ln\left(\liminf\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} = \sup_{0<t<1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} = \lim_{t\to 0} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} \tag{3.9}$$

and

$$M(w) = \inf_{t>1} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} = \lim_{t\to\infty} \frac{\ln\left(\limsup\limits_{h\to 0} \frac{w(ht)}{w(h)}\right)}{\ln t} \tag{3.10}$$

(see [90], [92]), [93], will be referred to as *the lower and upper indices* of the function $w(t)$. We have $0 \le m(w) \le M(w) \le \infty$ for $w \in W$.

The indices $m(\omega)$ and $M(\omega)$ may be also well defined for functions $w(t)$ positive for $t > 0$ which do not necessarily belong to W, for example, for $w \in \widetilde{W}$. Observe that

$$m(w_a) = a + m(w), \quad M(m_a) = a + M(w) \quad \text{where} \quad w_a(t) := t^a w(t) \tag{3.11}$$

and

$$m(w^\lambda) = \lambda m(w), \quad M(w^\lambda) = \lambda M(w), \quad \lambda \ge 0 \tag{3.12}$$

for every $w \in \widetilde{W}$.

The indices $m_\infty(w)$ and $M_\infty(w)$ responsible for the behavior of functions $w \in \Psi_\alpha^\beta([\ell,\infty))$ at infinity are introduced in the way similar to Definition in (3.9) and (3.10):

$$m_\infty(w) = \sup_{x>1} \frac{\ln\left[\liminf\limits_{h\to\infty} \frac{w(xh)}{w(h)}\right]}{\ln\ x}, \quad M_\infty(w) = \inf_{x>1} \frac{\ln\left[\limsup\limits_{h\to\infty} \frac{w(xh)}{w(h)}\right]}{\ln\ x}. \tag{3.13}$$

4. A SURVEY OF RECENT RESULTS ON BOUNDEDNESS OF CLASSICAL OPERATORS IN WEIGHTED SPACES $L^{P(\cdot)}(\Omega, \varrho)$

In this section we consider the following classical operators:

1) Convolution operators

$$Af(x) = \int\limits_{\mathbb{R}^n} k(y)f(x-y)dy$$

with rather "nice" kernels without log-condition,
2) Hardy-Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} |f(y)|\,d\mu(y), \quad x \in X \tag{4.1}$$

where X in general is a metric measure space, being either an open set in $\mathbb{R}^n$ or a Carleson curve on the complex plane in this section.
3) Calderón-Zygmund type singular integral operator

$$Tf(x) = \lim_{\varepsilon\to 0} \int\limits_{|x-y|>\varepsilon} k(x,y)f(y)\,dy \tag{4.2}$$

with the so called *standard kernel* (see, for instance, [24], p.99), and also the Cauchy singular integral

$$S_\Gamma f(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{f(\tau)}{\tau - t} d\nu(\tau) \tag{4.3}$$

along Carleson curves Γ on complex plane, where ν is the arc-length measure; we recall that Γ is called a *Carleson curve*, if it satisfies the condition

$$\nu(\Gamma \cap B(t,r)) \le Cr$$

where the constant $C > 0$ does not depend on $t \in \Gamma$ and $r > 0$;
4) potential type operators

$$I^{\alpha(\cdot)}f(x) \ = \int\limits_{\Omega} \frac{f(y)\,dy}{|x-y|^{n-\alpha(x)}}, \qquad 0 < \inf \alpha(x) \le \sup \alpha(x) < n \tag{4.4}$$

over open bounded sets Ω in $\mathbb{R}^n$, and
5) Hardy operators

$$H^\alpha f(x) = x^{\alpha-1}\int\limits_0^x \frac{f(y)}{y^\alpha}dy \qquad \text{and} \qquad \mathcal{H}_\beta f(x) = x^\beta \int\limits_x^\infty \frac{\varphi(y)\,dy}{y^{\beta+1}}. \tag{4.5}$$

Observe that boundedness of various classical operators in the non-weighted case was proved in [11] by the extrapolation method. In relation to the extrapolation method we refer also to [14], [15], [16].

4.1 On convolution operators

As is well known, the Young theorem in its natural form is not valid in the case of variable exponent, whatsoever smooth exponent $p(x)$ is. However, a natural expectation was that the Young theorem may be valid in the case of rather "nice" kernels and even without log-condition. This is true indeed, see Theorem 4.1 below.

By $\mathcal{P}_\infty(\mathbb{R}^n)$ we denote the set of measurable bounded functions $p : \mathbb{R}^n \to \mathbb{R}^1_+$ which satisfy the following conditions:

i) $1 \le p_- \le p(x) \le p_+ < \infty, \quad x \in \mathbb{R}^n$,

ii) there exists $p(\infty) = \lim\limits_{x\to\infty} p(x)$ and

$$|p(x) - p(\infty)| \le \frac{A}{\ln(2 + |x|)}, \quad x \in \mathbb{R}^n. \tag{4.6}$$

The following statement was proved in [23].

Theorem 4.1. *Let $k(y)$ satisfy the estimate*

$$|k(y)| \le \frac{C}{(1 + |y|)^\lambda}, \qquad y \in \mathbb{R}^n \tag{4.7}$$

for some $\lambda > n\left(1 - \frac{1}{p(\infty)} + \frac{1}{q(\infty)}\right)$. Then the convolution operator is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n)$ to the space $L^{q(\cdot)}(\mathbb{R}^n)$ under the only assumption that $p, q \in \mathcal{P}_\infty(\mathbb{R}^n)$ and $q(\infty) \ge p(\infty)$.

The convergence in $L^{p(\cdot)}(\mathbb{R}^n)$ convolutions with identity approximation kernels was studied in [10].

4.2 On maximal operator

Non-weighted boundedness of the maximal operator was first proved in [17], [18] for bounded domains or for $\mathbb{R}^n$ with $p(x) \equiv const$ outside some large ball. For further results in non-weighted case see [12], [13], [20], [76], [83], [84]. A special situation when $p(x)$ may tend to 1 or n, was studied in [9], [38], [47], [52]. For the non-validity of the modular inequality for the maximal function in case of non-constant $p(x)$ we refer to [77].

The result of [17], [18] was extended to the weighted case with power weights in [73]. After that several papers were devoted to consideration of more general weights for the variable exponent setting.

A characterization of general weights admissible for the boundedness of the maximal operator in the spirit of Muckenhoupt type condition is still an open question. The generalization to the case of more general weights encountered essential difficulties.

The main progress in obtaining sufficient conditions on weights, similar to those for constant p, in the Euclidean setting was obtained for a certain special class of weights, although essentially more general than the class of power weights. This class consists of weights of the form

$$\varrho(x) = [1 + w(|x|)] \prod_{k=1}^{m} w_k(|x - x_k|), \qquad x_k \in \overline{\Omega} \subset \mathbb{R}^n \tag{4.8}$$

of radial-type, the factor $1+w(|x|)$ appearing in the case of unbounded sets Ω. These weights have a typical feature of Muckenhoupt weights: they may oscillate between two power functions. This class may be also interpreted as a kind of Zygmund-Bary-Stechkin class. The introduction of this class of weights is based on the observation that the integral constructions involved in the Muckenhoupt condition for radial weights (in the case of constant p) are exactly those which appear in the Zygmund-type conditions.

For weights in this class it proved to be possible to obtain sufficient conditions of the boundedness of the maximal operator in terms of the so called upper and lower index numbers $m(w_k)$ and $M(w_k)$ of the weights $w_k(r)$ (similar in a sense to the Boyd indices). These conditions are obtained in the form of the natural numerical intervals

$$-\frac{n}{p(x_k)} < m(w_k) \le M(w_k) < \frac{n}{p'(x_k)} \tag{4.9}$$

"localized" to the nodes x_k of the weights $w_k(|x-x_0|)$. The sufficiency of this condition in terms of the numbers $m(w)$ and $M(w)$ is a new result even in the case of constant p. As is known, even in the case of constant p the verification of the Muckenhoupt condition for a concrete weight may be an uneasy task. Therefore, independently of finding an analogue of the Muckenhoupt condition for variable exponents, it is always of importance to find easy to check sufficient conditions for weight functions, as for instance in (4.9). The following theorem was proved in [66], see also a sketch of the proof in [74]. The Zygmund-Bary-Stechkin class Φ_n^0 and the notion of index numbers $m(w)$ and $M(w)$ are defined in Section 3..

Theorem 4.2. *Let $X=\Omega$ be a bounded domain in $\mathbb{R}^n$ with Lebesgue measure, let $p\in\mathbb{P}(\Omega)$ and ϱ be weight (4.8) with $w\equiv 0$. The operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(\Omega,\varrho)$, if $r^{\frac{n}{p(x_k)}}w_k(r)\in\Phi_n^0$ or, which is equivalent, conditions (4.9) are satisfied.*

In the case of power weights, a similar statement was proved in the context of Carleson curves Γ on complex planes, which are examples of metric measure spaces with arc length measure and coinciding upper and lower dimensions equal to $n=1$. Because of the interest to the case $X=\Gamma$ in the theory of singular integral equations, we formulate separately this statement proved in [69] (see also a sketch of the proof in [74]), including the case of infinite curves Γ.

In case $X=\Gamma$, for points on Γ we agree to write t instead of x, t_k instead of x_k, etc

Theorem 4.3. *Let Γ be a simple Carleson curve of finite or infinite length, let $p\in\mathbb{P}(\Gamma)$ and $p(t)\equiv p_\infty=const$ for $t\in\Gamma\backslash(\Gamma\cap B(0,R))$ for some $R>0$, if Γ is infinite. Then the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}(\Gamma,\varrho)$ with weight*

$$\varrho(t)=(1+|t|)^{\beta}\prod_{k=1}^{m}|t-t_k|^{\beta_k},\ t_k\in\Gamma, \tag{4.10}$$

if and only if

$$-\frac{1}{p(t_k)} < \beta_k < \frac{1}{p'(t_k)}, \qquad k = 1, ..., m, \qquad \text{and} \qquad -\frac{1}{p_\infty} < \beta + \sum_{k=1}^{m} \beta_k < \frac{1}{p'_\infty}, \tag{4.11}$$

the latter condition appearing in the case Γ *is infinite.*

We conclude this subsection by the observation that the fractional maximal operator for variable exponent setting was studied in [8], [11] and [70].

4.3 Weighted estimates for singular operators

For non-weighted estimates of singular integrals we refer to [22],

a) Calderón-Zygmund type operators. The following theorem on weighted boundedness of Calderón-Zygmund type singular operators (4.2) was proved in [68].

Theorem 4.4. *Let* Ω *be a bounded open set in* $\mathbb{R}^n$ *and* $p \in \mathbb{P}(\Omega)$. *A singular operator* T_Ω *with a standard kernel* $k(x, y)$, *bounded from* $L^1(\Omega)$ *to* $L^{1,\infty}(\Omega)$, *is bounded in the space* $L^{p(\cdot)}(\Omega, \varrho)$ *with the weight* $\varrho(x) = \prod_{k=1}^{m} w_k(|x - x_k|)$, *where* $x_k \in \overline{\Omega}$ *and*

$$w_k(r), \frac{1}{w_k(r)} \in \widetilde{W}([0, \ell]), \qquad k = 1, ..., m, \qquad \text{and} \qquad \ell = \operatorname{diam}\Omega, \tag{4.12}$$

if conditions (4.9) are satisfied.

Theorem 4.4 was obtained in [68] by means of the following general result, where

$$\mathcal{A}_{p(\cdot)}(\mathbb{R}^n) = \{\varrho : \text{ the maximal operator } \mathcal{M} \quad \text{is bounded in} \quad L^{p(\cdot)}(\mathbb{R}^n, \varrho)\} \tag{4.13}$$

Theorem 4.5. *([68]) Let* $p \in \mathbb{P}(\mathbb{R}^n)$*and let the weight function* ϱ *satisfy the assumptions*
i) $\varrho \in \mathcal{A}_{p(\cdot)}(\mathbb{R}^n)$ *and* $\frac{1}{\varrho} \in \mathcal{A}_{p'(\cdot)}(\mathbb{R}^n)$;
ii) there exists an $s \in (0, 1)$ *such that* $\frac{1}{\varrho^s} \in \mathcal{A}_{\left(\frac{p(\cdot)}{s}\right)'}(\mathbb{R}^n)$.
Then a singular operator T *with a standard kernel* $k(x, y)$ *and of weak (1,1)-type, is bounded in the space* $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$.

The non-weighted case $\varrho \equiv 1$ of Theorem 4.5 was proved in [22].

The following theorem was proved in [85] in the context adjusted to the theory of PDO for operators of the form

$$\mathbb{A}f(x) = \int_{\mathbb{R}^n} k(x, x - y) f(y) dy \tag{4.14}$$

where $k(x, z) \in C^1(\mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}))$ and it is assumed that

$$\lambda_1(\mathbb{A}) := \sup_{|\alpha|=1} \sup_{x,z \in \mathbb{R}^n \times \mathbb{R}^n} |z|^{n+1} |\partial_x^\alpha k(x, z)| < \infty, \tag{4.15}$$

$$\lambda_2(\mathbb{A}) := \sup_{|\beta|=1} \sup_{x,z\in\mathbb{R}^n\times\mathbb{R}^n} |z|^{n+1} \left|\partial_z^\beta k(x,z)\right| < \infty \tag{4.16}$$

and the operator $\mathbb{A}$ is of weak (1,1)-type:

$$|\{x \in \mathbb{R}^n : |\mathbb{A}f(x)| > t\}| \le \frac{C(\mathbb{A})}{t} \int\limits_{\mathbb{R}^n} |f(x)|\, dx. \tag{4.17}$$

We consider power weights

$$\varrho(x) = (1+|x|)^\beta \prod_{k=1}^m |x-x_k|^{\beta_k},\ x_k \in \mathbb{R}^n. \tag{4.18}$$

Theorem 4.6. *Let the operator $\mathbb{A}$ satisfy conditions (4.15)-(4.17).*
I. Let $p \in \mathbb{P}(\mathbb{R}^n)$ satisfy the decay condition

$$|p(x) - p(\infty)| \le \frac{A}{\ln{(2+|x|)}}, \qquad x \in \mathbb{R}^n, \tag{4.19}$$

Then the operator $\mathbb{A}$ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$.
II. Let $p \in \mathbb{P}(\mathbb{R}^n)$ be constant at infinity: $p(x) \equiv const = p_\infty$ for $|x| \ge R$ with some $R > 0$. Then the operator $\mathbb{A}$ is bounded in the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ with weight (4.18), if

$$-\frac{n}{p(x_k)} < \beta_k < \frac{n}{p'(x_k)}, \quad k = 1,...,m, \quad \textit{and} -\frac{n}{p_\infty} < \beta + \sum_{k=1}^m \beta_k < \frac{n}{p'_\infty}. \tag{4.20}$$

In both cases I and II

$$\|\mathbb{A}\|_{L^{p(\cdot)}(\mathbb{R}^n,\varrho)} \le c(n,p,\varrho)\, [\lambda_1(\mathbb{A}) + \lambda_2(\mathbb{A}) + C(\mathbb{A})] \tag{4.21}$$

where the constant $c(n,p,\varrho)$ depends only on n, p and ϱ.

For pseudo-differential operators

$$Au(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x,\xi) u(y) e^{i(x-y,\xi)} dy \tag{4.22}$$

we obtain the following corollary in which the known L. Hörmander class $S^0_{1,0}$ is the class of functions $a \in C^\infty\left(\mathbb{R}^n_x \times \mathbb{R}^n_\xi\right)$, such that

$$|a|_{r,t} = \sum_{|\alpha|\le r, |\beta|\le t} \sup_{\mathbb{R}^n\times\mathbb{R}^n} \left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \langle\xi\rangle^{|\alpha|} < \infty \tag{4.23}$$

for all the multi-indices α, β.

Corollary 4.7. *Let $p \in \mathbb{P}(\mathbb{R}^n)$ satisfy the decay condition (4.19). Then pseudo differential operators A with symbols $a(x,\xi)$ in the class $S^0_{1,0}$ are bounded in the space $L^{p(\cdot)}(\mathbb{R}^n)$.*

These results were used in [85] to establish criteria for Fredhomness of pseudo differential operators.

b) Cauchy singular integral operator. A theorem on the boundedness of the Cauchy singular operator in the variable exponent setting was first obtained for rather smooth curves, namely, Lyapunov curves or curves of bounded turning without cusps, see [72]. Meanwhile the modern development of the operator theory related to singular integral equations required a validity of such a boundedness on an arbitrary Carleson curve. The following result was proved in [65].

Theorem 4.8. *Let Γ be a simple Carleson curve of finite or infinite length, let $p \in \mathbb{P}(\Gamma)$ and the following condition at infinity*

$$|p(t) - p(\tau)| \leq \frac{A_\infty}{\ln \frac{1}{\left|\frac{1}{t} - \frac{1}{\tau}\right|}}, \qquad \left|\frac{1}{t} - \frac{1}{\tau}\right| \leq \frac{1}{2},$$

for $|t| \geq L$, $|\tau| \geq L$ with some $L > 0$, in the case Γ is an infinite curve. Then the singular operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma, \varrho)$ with weight (4.10), if and only if conditions (4.11) are satisfied.

An extension of Theorem 4.8 to the case of oscillating weights from the Zygmund-Bary-Stechkin class Φ_δ^β was given in [68].

We also mention the following boundedness result, obtained in [64], admitting a wide class of oscillating weights; it is an extension to the case of variable exponents of the known Helson-Szegö theorem [54]. To this end we need the notation

$$W^{p(\cdot)}(\Gamma) = \left\{ \varrho : \ \varrho S_\Gamma \frac{1}{\varrho} \quad \text{is bounded in} \quad L^{p(\cdot)}(\Gamma) \right\}.$$

Theorem 4.9. *Let Γ be a bounded Carleson curve and $p \in \mathbb{P}$, $\varrho \in W^{p(\cdot)}(\Gamma)$ and $\frac{1}{\varrho} \in L^{p'(\cdot)+\varepsilon}(\Gamma)$, where $\varepsilon > 0$. Then the function*

$$\varrho_\varphi(t) = \varrho(t) \exp \left| \frac{1}{2\pi} \int\limits_\Gamma \frac{\varphi(\tau)\, d\tau}{\tau - t} \right|$$

with real continuous φ, belongs to $W^{p(\cdot)}(\Gamma)$.

This statement, as formulated in Theorem 4.9, follows from Corollary 6.2 in [64], if we take into account that for a variable exponent $p \in \mathbb{P}$ the class of curves Γ for which the singular operator S_Γ is bounded in $L^{p(\cdot)}(\Gamma)$, coincides with the class of Carleson curves, as shows the following theorem, proved in [65].

Theorem 4.10. *Let Γ be a finite rectifiable curve. Let $p : \Gamma \to [1, \infty)$ be a continuous function. If the singular operator S_Γ is bounded in the space $L^{p(\cdot)}(\Gamma)$, then the curve Γ has the property*

$$\sup_{\substack{t \in \Gamma \\ r > 0}} \frac{\nu(\Gamma \cap B(t, r))}{r^{1-\varepsilon}} < \infty \tag{4.24}$$

for every $\varepsilon > 0$. If $p(t)$ satisfies the log-condition (2.3), then property (4.24) holds with $\varepsilon = 0$, that is, Γ is a Carleson curve.

From Theorem 4.8 the following statement important for applications is derived, see [65].

Theorem 4.11. *Let $a \in C(\Gamma)$ when Γ is a finite curve and $a \in C(\dot{\Gamma})$ when Γ is an infinite curve starting and ending at infinity, where $\dot{\Gamma}$ is the compactification of Γ by a single infinite point, that is, $a(t(-\infty)) = a(t(+\infty))$. Under the conditions of Theorem 4.8 the operator*

$$(S_\Gamma aI - aS_\Gamma)f = \frac{1}{\pi i}\int\limits_\Gamma \frac{a(\tau) - a(t)}{\tau - t} f(\tau)d\nu(\tau)$$

is compact in the space $L^{p(\cdot)}(\Gamma, \varrho)$.

In the Euclidean setting and non-weighted case, the compactness of the commutators generated by Calderón-Zygmund singular operators in $\mathbb{R}^n$ with $b \in BMO$ ($b \in VMO$ in the case of $\mathbb{R}^1$) was studied in [59] and [11]. The boundedness of multilinear commutators of singular operators on variable exponent Lebesgue spaces was studied in [104].

The results obtained for the Cauchy singular integral operator led to a possibility to investigate the Riemann problem in the setting of such spaces: find an analytic function Φ on the complex plane cut along Γ whose boundary values satisfy the conjugacy condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma, \tag{4.25}$$

where G and g are the given functions on Γ and Φ^+ and Φ^- are boundary values of Φ on Γ from inside and outside Γ, respectively. The solution of (4.25) is looked for in the class of analytic functions represented by the Cauchy type integral with density in the spaces $L^{p(\cdot)}(\Gamma)$ and it is assumed that g belongs to the same class. We refer to [41] and [82] for the classical solution of this problem and to [6], [43], [44], [62], [103] for L^p-solutions with constant p. The investigation of the effects generated by the variable exponent setting of the Riemann problem was given in [64], both for the case when the coefficient G is is continuous or piecewise continuous and also for the case of oscillating coefficient. The Fredholmness properties of the singular integral operators related to the Riemann problem in the spaces $L^{p(\cdot)}(\Gamma, \varrho)$ were studied in [71] and in more general setting in [57], [58].

4.4 On potential operators

For non-weighted results on potentials and Sobolev embeddings we refer to [11], [19], [25], [29], [32], [39], [81], [95].

a) Weighted $p(\cdot) \to q(\cdot)$-boundedness; the Euclidean case.

The known generalization of Sobolev theorem by Stein-Weiss for the case of power weights was extended in [100], [101] to the variable exponent setting as follows.

Theorem 4.12. *Let $p \in \mathbb{P}(\mathbb{R}^n)$, $\sup_{x\in\mathbb{R}^n} p(x) < \frac{n}{\alpha}$, $\varrho(x) = |x|^{\gamma_0}(1+|x|)^{\gamma_\infty - \gamma_0}$ and*

$$|p_*(x)-p_*(y)| \le \frac{A_\infty}{\ln \frac{1}{|x-y|}}, \qquad |x-y| \le \frac{1}{2}, \ \ x,\, y \in \mathbb{R}^n, \qquad p_*(x) = p\left(\frac{x}{|x|^2}\right), \tag{4.26}$$

the operator

$$I^\alpha f(x) \ = \int\limits_{\mathbb{R}^n} \frac{f(y)\, dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n \tag{4.27}$$

is bounded from the space $L^{p(\cdot)}(\mathbb{R}^n, \varrho)$ into the space $L^{q(\cdot)}(\mathbb{R}^n, \varrho)$, if

$$\alpha - \frac{n}{p(0)} < \gamma_0 < \frac{n}{p'(0)}, \qquad \alpha - \frac{n}{p(\infty)} < \gamma_\infty < \frac{n}{p'(\infty)}. \tag{4.28}$$

Recently results similar to Theorem 4.12 were obtained for more general weights like in Theorem 4.2.

We give separate formulations of this generalizations for bounded domains and for the whole space $\mathbb{R}^n$ because for bounded domains we are also able to admit variable orders α of the potential operator.

We assume that

$$0 < \inf_{x\in\Omega} \alpha(x)p(x) \le \sup_{x\in\Omega} \alpha(x)p(x) < n \tag{4.29}$$

and

$$|\alpha(x) - \alpha(y)| \le \frac{A}{\ln \frac{1}{|x-y|}}, \qquad x, y \in \Omega, \qquad |x-y| \le \frac{1}{2}. \tag{4.30}$$

Theorem 4.13. *Let Ω be a bounded open set in $\mathbb{R}^n$ and $x_0 \in \overline{\Omega}$, let $p \in \mathbb{P}(\Omega)$ and α satisfy conditions (4.29)-(4.30). Let also $\varrho(x) = w(|x-x_0|), x_0 \in \overline{\Omega}$, where*

$$w(r) \in \Phi^\beta_\gamma([0,\ell]) \quad \textit{with} \quad \beta = \alpha(x_0) - \frac{n}{p(x_0)}, \quad \gamma = \frac{n}{p'(x_0)}, \tag{4.31}$$

or equivalently

$$w \in \widetilde{W}_0 \quad \textit{and} \quad \alpha(x_0) - \frac{n}{p(x_0)} < m(w) \le M(w) < \frac{n}{p'(x_0)}. \tag{4.32}$$

Then

$$\left\| I^{\alpha(\cdot)} f \right\|_{L^{q(\cdot)}(\Omega,\varrho)} \le C \, \|f\|_{L^{p(\cdot)}(\Omega,\varrho)} \, . \tag{4.33}$$

$w_0(r)$ belongs to some Φ^β_γ-class on $[0,1]$ and $w_\infty(r)$ belongs to some Ψ^β_γ-class on $[1,\infty]$ and both the weights are continued by constant to $[0,\infty)$:

$$w_0(r) \equiv w_0(1), \quad 1 \le r < \infty \quad \text{and} \quad w_\infty(r) \equiv w_\infty(1), \quad 0 < r \le 1.$$

Theorem 4.14. *Let $0 < \alpha < n$, $p \in \mathbb{P}(\mathbb{R}^n)$ satisfy assumption (4.26) and condition $p^+(\mathbb{R}^n) < \frac{n}{\alpha}$, and let $\varrho(x) = w_0(|x|)w_\infty(|x|)$. The operator I^α is bounded* *from $L^{p(\cdot)}(\mathbb{R}^n,\varrho)$ to $L^{q(\cdot)}(\mathbb{R}^n,\varrho)$, if*

$$w_0(r)\in\Phi^{\beta_0}_{\gamma_0}([0,1]),\quad w_\infty(r)\in\Psi^{\beta_\infty}_{\gamma_\infty}([1,\infty)) \tag{4.34}$$

where $\beta_0=\alpha-\frac{n}{p(0)}$, $\gamma_0=\frac{n}{p'(0)}$, $\beta_\infty=\frac{n}{p'(\infty)}$, $\gamma_\infty=\alpha-\frac{n}{p(\infty)}$, or equivalently

$$w_0\in\widetilde{W}([0,1]),\quad \alpha-\frac{n}{p(0)}<m(w_0)\le M(w_0)<\frac{n}{p'(0)}, \tag{4.35}$$

and

$$w_\infty\in\widetilde{W}([1,\infty]),\quad \alpha-\frac{n}{p(\infty)}<m(w_\infty)\le M(w_\infty)<\frac{n}{p'(\infty)}. \tag{4.36}$$

We refer also to the paper [26] where there was obtained an extension of the Adams' trace inequality for potentials to the variable exponent setting on homogeneous spaces.

b) Weighted $p(\cdot)\to q(\cdot)$-boundedness for potentials on Carleson curves. A statement on $p(\cdot)\to q(\cdot)$ boundedness of potential operators

$$I^{\alpha(\cdot)}f(t)\;=\;\int\limits_\Gamma\frac{f(\tau)\,d\nu(\tau)}{|t-\tau|^{1-\alpha(t)}} \tag{4.37}$$

is also valid on an arbitrary Carleson curve Γ and for variable orders $\alpha(t)$ as well, see [69], as given below.

Theorem 4.15. *Let Γ be a simple Carleson curve of a finite length, $p\in\mathbb{P}(\Gamma)$ and $\alpha(t)$ satisfy the assumptions*

$$0<\inf_{t\in\Gamma}\alpha(t)p(t)\le\sup_{t\in\Gamma}\alpha(t)p(t)<1. \tag{4.38}$$

Then the operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}(\Gamma)$ into the space $L^{q(\cdot)}(\Gamma)$ with $\frac{1}{q(t)}=\frac{1}{p(t)}-\alpha(t)$. This statement remains valid for infinite Carleson curves if, in addition to the above conditions, $p(t)=p_\infty=const$ and $\alpha(t)=\alpha_\infty=const$ outside some circle $B(t_0,R),t_0\in\Gamma$.

In the next weighted version of Theorem 4.15 for finite curves we additionally suppose that the order $\alpha(t)$ is log-continuous at the nodes of the weight:

$$|\alpha(t)-\alpha(t_k)|\le\frac{A}{|\ln|t-t_k||},\quad k=1,...,m. \tag{4.39}$$

Theorem 4.16. *Let Γ be a simple Carleson curve of a finite length. Under condition (4.39) and the assumptions of Theorem 4.15, the operator $I^{\alpha(\cdot)}$ is bounded from the space $L^{p(\cdot)}(\Gamma,\varrho)$ into the space $L^{q(\cdot)}(\Gamma,\varrho)$ where $\frac{1}{q(t)}=\frac{1}{p(t)}-\alpha(t)$, and the weight $\varrho(t)=\prod\limits_{k=1}^m|t-t_k|^\beta_k$, if*

$$\alpha(t_k)-\frac{1}{p(t_k)}<\beta_k<1-\frac{1}{p(t_k)},\qquad k=1,...,m. \tag{4.40}$$

Corollary 4.17. *Under the assumptions of Theorem 4.16, the fractional maximal operator*

$$M_{\alpha(\cdot)}f(t) = \sup_{r>0} \frac{1}{\nu\{\Gamma(t,r)\}^{n-\alpha(t)}} \int\limits_{\Gamma(t,r)} |f(\tau)|\, d\nu(\tau), \qquad \Gamma(t,r) = \Gamma \cap B(t,r)$$

is bounded from the space $L^{p(\cdot)}(\Gamma,\varrho)$ into the space $L^{q(\cdot)}(\Gamma,\varrho)$.

We mention also Hardy-type inequalities for potentials obtained in [97], [98] in the multidimensional case and in [28] in the one-dimensional case, including in particular the case of Riemann-Liouville and Weyl fractional integrals.

c) Characterization of the range of potential operators. The inversion of the Riesz potentials with densities in $L^{p(\cdot)}(\mathbb{R}^n)$ by means of hypersingular integrals $\mathbb{D}^\alpha f$ (Riesz fractional derivatives of order α) was obtained in [1] (we refer to [96] for the case of constant p and hypersingular integrals in general). This gave a possibility to give in [2] a characterization of the range $I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$ in terms of convergence of $\mathbb{D}^\alpha f$ in $L^{p(\cdot)}(\mathbb{R}^n)$ as follows.

Theorem 4.18. *Let $p \in \mathbb{P}(\mathbb{R}^n)$, $0 < \alpha < n$, $1 < p_-(\mathbb{R}^n) \le p^+(\mathbb{R}^n) < \frac{n}{\alpha}$ and let f be a locally integrable function. Then $f \in I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)]$, if and only if $f \in L^{q(\cdot)}(\mathbb{R}^n)$, with $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, and there exists the Riesz derivative $\mathbb{D}^\alpha f$ (in the sense of convergence in $L_{p(\cdot)}$).*

A study of the range $I^\alpha[L^{p(\cdot)}(\Omega)]$ for domains $\Omega \subset \mathbb{R}^n$ is an open question; in the form given in Theorem 4.18 it is open even in the case of constant p, one of the reasons being in the absence of the corresponding apparatus of hypersingular integrals adjusted to domains in $\mathbb{R}^n$; some their analogue reflecting the influence of the boundary was recently suggested in [86] for the case $0 < \alpha < 1$. In the one dimensional case for $\Omega = (a,b), -\infty < a < b \le \infty$, when the range of the potential coincides with that of the Riemann-Liouville fractional integral operators (in the case $1 < p^+(a,b) < \frac{1}{\alpha}$), the characterization for variable $p(x)$ was obtained in [87], where for $-\infty < a < b < \infty$ there was also shown its coincidence with the space of restrictions of Bessel potentials.

The result of Theorem 4.18 was used in [2] to obtain a characterization of the *Bessel potential space*

$$\mathcal{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = \{f : \ f = \mathcal{B}^\alpha\varphi, \quad \varphi \in L^{p(\cdot)}(\mathbb{R}^n)\}, \quad \alpha \ge 0,$$

where $\mathcal{B}^\alpha\varphi = F^{-1}(1+|\xi|^2)^{-\alpha/2}F\varphi$ and runs as follows.

Theorem 4.19. *Under the conditions of Theorem 4.18*

$$\mathcal{B}^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = L^{p(\cdot)}(\mathbb{R}^n) \bigcap I^\alpha[L^{p(\cdot)}(\mathbb{R}^n)] = \{f \in L^{p(\cdot)}(\mathbb{R}^n) : \ \mathbb{D}^\alpha f \in L^{p(\cdot)}(\mathbb{R}^n)\} \tag{4.41}$$

and $\mathcal{B}^m[L^{p(\cdot)}(\mathbb{R}^n)] = W^{m,p(\cdot)}(\mathbb{R}^n)$ for any integer $m \in \mathbb{N}_0$, where $W^{m,p(\cdot)}(\mathbb{R}^n)$ is the Sobolev space with the variable exponent $p(x)$.

Statement (4.41) has the following generalization, see [87], Theorem 4.10. (We refer to [5] for the notion of Banach function spaces).

Theorem 4.20. *Let* $\mathrm{Y} = \mathrm{Y}(\mathbb{R}^n)$ *be a Banach function space, satisfying the assumptions*
i) C_0^∞ *is dense in* Y*;*
ii) the maximal operator $\mathcal{M}$ *is bounded in* Y*;*
iii) $I^\alpha f(x)$ *converges absolutely for almost all* x *for every* $f \in \mathrm{Y}$ *and* $(1+|x|)^{-n-\alpha} I^\alpha f(x) \in L^1(\mathbb{R}^n)$.
Then

$$\mathcal{B}^\alpha(\mathrm{Y}) = \mathrm{Y} \bigcap I^\alpha(\mathrm{Y}) = \{f \in \mathrm{Y} : \mathbb{D}^\alpha f = \lim_{\substack{\varepsilon \to 0 \\ (\mathrm{Y})}} \mathbb{D}^\alpha_\varepsilon f \in \mathrm{Y}\}. \tag{4.42}$$

From Theorem 4.20 there follows, in particular, the characterization of the ranges of potential operators over weighted Lebesgue spaces with variable exponent obtained by means of results of Subsection 4.2 for the maximal operator.

Observe that certain results related to imbedding of the range of the Riesz potential operator into Hölder spaces (of variable order) in the case $p(x) \geq n$ were obtained in [3]. The results proved in [3] run as follows where

$$\Pi_{p,\Omega} := \{x \in \Omega : p(x) > n\} \tag{4.43}$$

and $C^{0,\alpha(\cdot)}(\Omega)$ is the space of bounded continuous functions f with a finite seminorm

$$[f]_{\alpha(\cdot),\Omega} := \sup_{\substack{x, x+h \in \Omega \\ 0<|h|\leq 1}} \frac{|f(x+h) - f(x)|}{|h|^{\alpha(x)}}.$$

Theorem 4.21. *Let* Ω *be a bounded open set with Lipschitz boundary and let* $p(\cdot)$ *satisfy the log-condition (2.3) and have a non-empty set* $\Pi_{p,\Omega}$*. If* $f \in W^{1,p(\cdot)}(\Omega)$*, then*

$$|f(x) - f(y)| \leq C(x,y)\, \||\nabla f|\|_{p(\cdot),\Omega}\, |x-y|^{1-\frac{n}{\min[p(x),p(y)]}} \tag{4.44}$$

for all $x, y \in \Pi_{p,\Omega}$ *such that* $|x-y| \leq 1$*, where*

$$C(x,y) = \frac{c}{\min[p(x),p(y)] - n}$$

with $c > 0$ *not depending on* f, x *and* y.

Theorem 4.22. *Let* Ω *be a bounded open set with Lipschitz boundary and suppose that* $p(\cdot)$ *satisfies the logarithmic condition (2.3). If* $\inf_{x\in\Omega} p(x) > n$*, then*

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow C^{0,1-\frac{n}{p(\cdot)}}(\Omega), \tag{4.45}$$

where “$\hookrightarrow$*” means continuous embedding.*

Theorem 4.22 is an improved version of the result earlier obtained in [29], [34]. The papers [36], [37] are also relevant to the topic. We refer also to [46] where the capacity approach was used to get embeddings into the space of continuous functions or into $L^\infty(\Omega)$.

Observe that is [3] there were also obtained $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}$-estimates of hypersingular integrals (fractional differentiation operators)

$$\mathcal{D}^{\alpha(\cdot)} f(x) = \int_\Omega \frac{f(x) - f(y)}{|x-y|^{n+\alpha(x)}}\, dy, \quad x \in \Omega. \tag{4.46}$$

4.5 On Hardy operators

For Hardy operators (4.5) in [23] the following result was obtained without the log condition at all points on $\mathbb{R}^1_+$. By $\mathcal{M}_{0,\infty}(\mathbb{R}^1_+)$ we denote the set of all measurable bounded functions $p(x) : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ which satisfy the following conditions:

i) $\quad 0 \le p_- \le p(x) \le p_+ < \infty, \quad x \in \mathbb{R}^1_+,$

ii_0) there exists $p(0) = \lim\limits_{x\to 0} p(x)$ and $|p(x) - p(0)| \le \frac{A}{\ln \frac{1}{x}}, \quad 0 < x \le \frac{1}{2},$

ii_∞) there exists $\mu(\infty) = \lim\limits_{x\to\infty} p(x)$ and $|p(x) - p(\infty)| \le \frac{A}{\ln x}, \quad x \ge 2.$

By $\mathcal{P}_{0,\infty} = \mathcal{P}_{0,\infty}(\mathbb{R}^1_+)$ we denote the subset of functions $p(x) \in \mathcal{M}_{0,\infty}(\mathbb{R}^1_+)$ with $\inf\limits_{x\in\mathbb{R}^1_+} p(x) \ge 1$.

Theorem 4.23. *Let $p, q \in \mathcal{P}_{0,\infty}(\mathbb{R}^1_+)$ and $\mu \in \mathcal{M}_{0,\infty}(\mathbb{R}^1_+)$ and let*

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0), \ \frac{1}{q(\infty)} = \frac{1}{p(\infty)} - \mu(\infty) \ \text{ and } \ 0 \le \mu(0) < \frac{1}{p(0)}, \ 0 \le \mu(\infty) < \frac{1}{p(\infty)}.$$

Then the Hardy-type inequalities

$$\left\| x^{\alpha+\mu(x)-1} \int_0^x \frac{f(y)\,dy}{y^\alpha} \right\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \le C \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)} \tag{4.47}$$

and

$$\left\| x^{\beta+\mu(x)} \int_x^\infty \frac{f(y)\,dy}{y^{\beta+1}} \right\|_{L^{q(\cdot)}(\mathbb{R}^1_+)} \le C \|f\|_{L^{p(\cdot)}(\mathbb{R}^1_+)}, \tag{4.48}$$

are valid, if and only if

$$\alpha < \min\left\{ \frac{1}{p'(0)}, \frac{1}{p'(\infty)} \right\} \quad \text{and} \quad \beta > \max\left\{ \frac{1}{p(0)}, \frac{1}{p(\infty)} \right\}. \tag{4.49}$$

For previous version of Hardy inequality we refer to [48], [73]. In [48] a multidimensional analogue of Hardy inequality was also considered.

5. WEIGHTED BOUNDEDNESS OF MAXIMAL OPERATORS ON METRIC MEASURE SPACES

In the case of constant $p \in (1, \infty)$ the boundedness of the maximal operator on bounded metric measure spaces is well known, due to A.P.Calderón [7] and R.Macías and C.Segovia [78] for weights in the Muckenhoupt class $\mathcal{A}_p = \mathcal{A}_p(X)$, defined by the condition

$$\sup_{x\in X, r>0} \left(\frac{1}{\mu B(x,r)} \int_{B(x,r)} |\varrho(y)|^p d\mu(y) \right) \left(\frac{1}{\mu B(x,r)} \int_{B(x,r)} \frac{d\mu(y)}{|\varrho(y)|^{p'}} \right)^{p-1} < \infty. \tag{5.1}$$

For variable exponents the maximal operator on metric measure spaces was considered in [50] and [61], where the following non-weighted result was obtained.

Theorem 5.1. *Let a bounded metric measure space X satisfy the doubling condition (2.1) and $p \in \mathbb{P}(X)$. Then the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}(X)$.*

As was observed in [50], in contrast to the case of constant p, the doubling condition is not necessary for the boundedness of the maximal operator when p is variable.

The boundedness of the operator $\mathcal{M}$ in the weighted space $L^{p(\cdot)}(X,\varrho)$ is known for the cases where $X = \Omega$ is a bounded domain in $\mathbb{R}^n$ or $X = \Gamma$ is a Carleson curve on the complex plane, see Theorems 4.2 and 4.3.

For an arbitrary metric measure space with doubling condition, we present in this section new results on weighted boundedness given in Theorems A, B and C stated below. There proof taking too much space will be given elsewhere.

Let $\mathcal{A}_{p(\cdot)}(X)$ be the class (4.13). To formulate Theorem A we introduce the following "Muckenhoupt-like looking" class $\widetilde{\mathcal{A}}_{p(\cdot)}(X)$ of weights, which satisfy the condition

$$\sup_{x\in X, r>0}\left(\frac{1}{\mu B(x,r)}\int\limits_{B(x,r)}|\varrho(y)|^{p(y)}d\mu(y)\right)\left(\frac{1}{\mu B(x,r)}\int\limits_{B(x,r)}\frac{d\mu(y)}{|\varrho(y)|^{\frac{p(y)}{p_--1}}}\right)^{p_--1}<\infty. \quad (5.2)$$

This class $\widetilde{\mathcal{A}}_{p(\cdot)}(X)$ used in Theorem A is narrower than the class $\mathcal{A}_{p(\cdot)}$. However, it coincides with the Muckenhoupt class $\mathcal{A}_p$ in case p is constant.

In Theorem A, under log-condition on p and doubling condition on the measure we show that

$$\widetilde{\mathcal{A}}_{p(\cdot)}(X) \subset \mathcal{A}_{p(\cdot)}(X).$$

In Theorem B we deal with a special class of radial type weights in the Zygmund-Bary-Stechkin class and arrive at the necessity to relate the properties of the weight to those of the measure $\mu B(x,r)$ as stated in (1.1). Such a result for the Euclidean case was earlier obtained in [66]. The proof for the case of metric measure spaces requires an essential modification of the technique used. Theorem B is proved by means of Theorem A, but it is not contained in Theorem A, being more general in its range of applicability.

Theorem A. *Let X be a bounded doubling metric measure space, let the exponent $p \in \mathbb{P}(X)$ and the weight ϱ fulfill condition (5.2). Then the operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(X,\varrho)$.*

In Theorems B and C we deal with bounded and unbounded metric spaces, respectively. In Theorem B we consider weights of the form

$$\varrho(x) = \prod_{k=1}^{N} w_k(d(x, x_k)), \quad x_k \in X, \tag{5.3}$$

where x_k are distinct points and $w_k(r)$ may oscillate between two power functions as $r \to 0+$ (radial Zygmund-Bary-Stechkin type weights), and in Theorem C we consider similar weights of the form

$$\varrho(x) = w_0[1 + d(x_0, x)] \prod_{k=1}^{N} w_k[d(x, x_k)], \quad x_k \in X, k = 0, 1, ...N. \tag{5.4}$$

We make also use of the following numbers which play a role of dimensions of the space (X, d, μ) at the point $x \in X$:
1) the *local lower and upper dimensions*

$$m(\mu B_x) = \sup_{t>1} \frac{\ln\left(\liminf\limits_{r\to 0} \frac{\mu B(x,rt)}{\mu B(x,r)}\right)}{\ln t}, \quad M(\mu B_x) = \inf_{t>1} \frac{\ln\left(\limsup\limits_{r\to 0} \frac{\mu B(x,rt)}{\mu B(x,r)}\right)}{\ln t}, \tag{5.5}$$

2) *similar dimensions ”influenced” by infinity*:

$$m_\infty(\mu B) = \sup_{t>1} \frac{\ln\left(\liminf\limits_{r\to \infty} \frac{\mu B(x,rt)}{\mu B(x,r)}\right)}{\ln t}, \quad M_\infty(\mu B) = \inf_{t>1} \frac{\ln\left(\limsup\limits_{r\to \infty} \frac{\mu B(x,rt)}{\mu B(x,r)}\right)}{\ln t} \tag{5.6}$$

the latter appearing only in the case of unbounded X. The idea to use the above local dimensions was borrowed from paper [89]. It may be shown that the numbers $m_\infty(\mu B)$ and $M_\infty(\mu B)$ do not depend on x, see [89].

Remark 5.2. In a different form local dimensions were introduced and/or used in [30], [31], [49], [50], [55]. The introduction of the local dimensions in the form described above was influenced by the study of lower and upper indices of oscillating almost increasing functions in [90], [92], [93] and application of that study in [67], [66], [68], [89], [94].

It may be shown (see [94]) that for an arbitrarily small $\varepsilon > 0$

$$c_1 r^{M(\mu B_x)+\varepsilon} \le \mu B(x, r) \le c_1 r^{m(\mu B_x)-\varepsilon}, \quad 0 < r \le R < \infty \tag{5.7}$$

and

$$c_3 r^{m_\infty(\mu B)-\varepsilon} \le \mu B(x, r) \le c_4 r^{M_\infty(\mu B)+\varepsilon}, \quad r_0 \le r < \infty, \tag{5.8}$$

where $c_i, i = 1, 2, 3, 4$, depend on $\varepsilon > 0$, but do not depend on r and x.

In the sequel we will use the ”uniform” index $m(\mu B)$ introduced in (1.2).

The Zygmund-Bary-Stechkin class Φ_1^0 of weights and the upper and lower indices of weights (of the type of Matuszewska-Orlicz indices, see [79], close in a sense to the Boyd indices) used in the theorem below, were defined in Section 3.. Various non-trivial examples of functions in Zygmund-Bary-Stechkin-type classes with coinciding indices may be found in [90], Section II; [91], Section 2.1, and with non-coinciding indices in [93].

Theorem B. *Let X be a bounded doubling metric measure space and let $p \in \mathbb{P}(X)$. The operator $\mathcal{M}$ is bounded in $L^{p(\cdot)}(X, \varrho)$ with weight (5.3), if $r^{\frac{m(\mu B)}{p(x_k)}} w_k(r) \in \Phi^0_{m(\mu B)}$*, or equivalently $w_k \in \widetilde{W}([0,\ell])$, $\ell =$ diam X, and

$$-\frac{m(\mu B)}{p(x_k)} < m(w_k) \leq M(w_k) < \frac{m(\mu B)}{p'(x_k)} \quad , \quad k = 1, 2, ..., N. \tag{5.9}$$

In the case where X is a bounded open set in $\mathbb{R}^n$, Theorem B was proved in [66] and coincides with Theorem 4.2; in the case where $X = \Gamma$ is a Carleson curve, it was proved in [69] for power weights, as stated in Theorem 4.3, and in [67] for weights in Zygmund-Bary-Stechkin type class.

Theorem C. *Let X be an unbounded doubling metric measure space and let $p \in \mathbb{P}(X)$, and let there exist a ball $B(x_0, R), x_0 \in X$ such that $p(x) \equiv p_\infty = const$ for $x \in X \backslash B(x_0, R)$. Then the maximal operator $\mathcal{M}$ is bounded in the space $L^{p(\cdot)}(X, \varrho)$, with weight (5.4), if $w_k \in \widetilde{W}(\mathbb{R}^1_+)$ and*

$$-\frac{m(\mu B)}{p(x_k)} < m(w_k) \leq M(w_k) < \frac{m(\mu B)}{p'(x_k)}, \ \ k = 1, ..., N, \tag{5.10}$$

and

$$-\frac{m_\infty(\mu B)}{p_\infty} < \sum_{k=0}^{N} m_\infty(w_k) \leq \sum_{k=0}^{N} M_\infty(w_k) < \frac{m_\infty(\mu B)}{p'_\infty} - \Delta_{p_\infty}, \tag{5.11}$$

where $\Delta_{p_\infty} = \frac{M_\infty(\mu B) - m_\infty(\mu B)}{p_\infty}$.

In particular, for the power type weight

$$\varrho(x) = (1 + d(x_0, x))^{\beta_0} \prod_{k=1}^{N} [d(x, x_k)]^{\beta_k}, \quad x_k \in X, k = 0, 1, ...N \tag{5.12}$$

conditions (5.10)-(5.11) take the form

$$-\frac{m(\mu B)}{p(x_k)} < \beta_k < \frac{m(\mu B)}{p'(x_k)}, \quad k = 1, ..., N, \tag{5.13}$$

and

$$-\frac{m_\infty(\mu B)}{p_\infty} < \sum_{k=0}^{N} \beta_k < \frac{m_\infty(\mu B)}{p'_\infty} - \Delta_{p_\infty}. \tag{5.14}$$

The bounds in (5.11) turn to take a natural form $-\frac{m_\infty(\mu B)}{p_\infty}$ and $\frac{m_\infty(\mu B)}{p'_\infty}$ with $\Delta_{p_\infty} = 0$ when "dimensions" $m_\infty(\mu B)$ and $M_\infty(\mu B)$ coincide with each other. In particular, in the case where X has a constant dimension $d > 0$ in the sense that

$$C_1 r^d \leq \mu B(x, r) \leq C_2 r^d,$$

conditions (5.13)-(5.14) take the form

$$-\frac{d}{p(x_k)} < \beta_k < \frac{d}{p'(x_k)}, \qquad k = 1, ..., N, \quad -\frac{d}{p_\infty} < \sum_{k=0}^{N} \beta_k < \frac{d}{p'_\infty}. \tag{5.15}$$

The Euclidean space version of Theorem C for variable exponents and power weights was obtained in [60].

It goes without saying that in Theorems B and C we should be interested in the result with the values $m(\mu B_{x_k})$ instead of taking a kind of the infinum in x of $m(\mu B_x)$ with respect to all x. However, such a localization of the index $m(\mu B_x)$ remains an open question. So, instead, we deal with that infimum which, of course, serves for all the points x_k.

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REFERENCES

1. Almeida, A. (2003). Inversion of the Riesz Potential Operator on Lebesgue Spaces with Variable Exponent. *Frac. Calc. Appl. Anal.*, **6(3)**, 311–327.
2. Almeida, A. and Samko, S. (2006). Characterization of Riesz and Bessel potentials on variable Lebesgue spaces. *J. Function Spaces and Applic.*, **4(2)**, 113–144.
3. Almeida, A. and Samko, S. (2007). Pointwise inequalities in variable Sobolev spaces and applications. *Zeit. Anal. und ihre Anwend.*, **26(2)**, 179–193.
4. Bari, N. K. and Stechkin, S. B. (1956). Best approximations and differential properties of two conjugate functions. *Proceedings of Moscow Math. Soc.*, **5**, 483–522 (in Russian).
5. Bennett, C. and Sharpley, R. (1988). *Interpolation of Operators.*, volume **129** of *Pure and Applied Mathematics.* Academic Press Inc., Boston, MA.
6. Böttcher, A. and Karlovich, Yu. (1997). *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators.* Basel, Boston, Berlin: Birkhäuser Verlag, 397 pages.
7. Calderón, A.-P. (1976). Inequalities for the maximal function relative to a metric. *Studia Math.*, **57(3)**, 297–306.
8. Capone, C., Cruz-Uribe, D. and Fiorenza, A. (2004). The fractional maximal operator on variable L_p spaces. *Preprint of Istituto per le Applicazioni del Calcolo "Mauro Picone" - Sezione di Napoli.*, (RT **281/04**), 1–22.
9. Cruz-Uribe, D. and Fiorenza, A. (2006). *LLogL* results for the maximal operator in variableL^p spaces. *Preprint, 326/06, http://www.na.iac.cnr.it/rapporti/anno*2006*.htm*, 22 pages.
10. Cruz-Uribe, D. and Fiorenza, A. (2007). Approximate identities in variable L^p spaces. *Math. Nachr.*, **280**, 256–270.

11. Cruz-Uribe, D., Fiorenza, A., Martell, J. M. and Perez C. (2006). The boundedness of classical operators on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.*, **31(1)**, 239–264.
12. Cruz-Uribe, D., Fiorenza, A. and Neugebauer, C. J. (2003). The maximal function on variable L^p-spaces. *Ann. Acad. Scient. Fennicae, Math.*, **28**, 223–238.
13. Cruz-Uribe, D., Fiorenza, A. and Neugebauer, C. J. (2004). Corrections to "The maximal function on variable L^p-spaces". *Ann. Acad. Scient. Fennicae, Math.*, **29**, 247–249.
14. Cruz-Uribe, D., Martell, J. M. and Pérez, C. (2004). Extrapolation from A_∞ weights and applications. *J. Funct. Anal.*, **213(2)**, 412–439.
15. Cruz-Uribe, D., Martell, J. M. and Pérez, C. (2006). Extensions of Rubio de Francia's extrapolation theorem. *Collect. Math.*, (Vol. Extra), 195–231.
16. Curbera, G. P., García-Cuerva, J., Martell, J. M. and Pérez, C. (2006). Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals. *Adv. Math.*, **203(1)**, 256–318.
17. Diening, L. (2002). Theoretical and numerical results for electrorheological fluids. Ph.D Thesis, University of Freiburg, Germany.
18. Diening, L. (2004). Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.*, **7(2)**, 245–253.
19. Diening, L. (2004). Riesz potential and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. *Mathem. Nachrichten*, **268**, 31–43.
20. Diening, L. (2005). Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. *Bull. Sci. Math.*, **129(8)**, 657–700.
21. Diening, L., Hästö, P. and Nekvinda, A. (2004). Open problems in variable exponent Lebesgue and Sobolev spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 - June 2, 2004*. Math. Inst. Acad. Sci. Czech Republick, Praha.
22. Diening, L. and Ružička, M. (2003). Calderon-Zygmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamics. *J. Reine Angew. Math*, **563**, 197–220.
23. Diening, L. and Samko, S. (2006). Hardy inequality in variable exponent Lebesgue spaces. Albert-Ludwigs-Universität Freiburg. Preprint Nr. 1/2006-17.03.2006, 15p.
24. Duoandikoetxea, J. (2001). *Fourier Analysis.* Amer. Math. Soc., "Graduate Studies", vol. **29**.
25. Edmunds, D. E. (2002). Sobolev embeddings with variable exponent, II. *Math. Nachr.*, **246/247**, 53–67.
26. Edmunds, D. E., Kokilashvili, V. and Meskhi, A. (2004). A trace inequality for generalized potentials in Lebesgue spaces with variable exponent. *J. Function Spaces and Appl.*, **q(1)**, 55–69.
27. Edmunds, D. E., Lang, J. and Nekvinda, A. (1999). On $L^{p(x)}$ norms. *Proc. R. Soc. Lond.*, A **455**, 219–225.
28. Edmunds, D. E. and Meskhi, A. (2002). Potential Type Operators in and $L^{p(x)}$ Spaces. *Zeitschrift für Analysis und ihre Anwendungen*, **21(3)**, 681–690.
29. Edmunds, D. E. and Rákosník, J. (2000). Sobolev embeddings with variable exponent. *Studia Math.*, **143(3)**, 267–293.
30. Falconer, K. (1997). *Techniques in fractal geometry.* John Wiley & Sons Ltd., Chichester.
31. Falconer, K. J. and O'Neil, T. C. (1999). Convolutions and the geometry of multifractal measures. *Math. Nachr.*, **204**, 61–82.
32. Fan, X., Shen, J. and Zhao, D. (2001). Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, **262(2)**, 749–760.

33. Fan, X. and Zhao, D. (2001). On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. *J. Math. Anal. Appl.*, **263(2)**, 424–446.
34. Fiorenza, A. (2002). A mean continuity type result for certain Sobolev spaces with variable exponent. *Commun. Contemp. Math.*, **4(3)**, 587–605.
35. Futamura, T., Harjulehto, P., Hästö, P., Mizuta, Y. and Shimomura, T. (2005). Variable exponent spaces on metric measure spaces. *Proceedings of the 5th ISAAC Congress, Catania, 2005.* to appear; available at *http : //www.helsinki.fi/ pharjule/varsob/publications.shtml.*
36. Futamura, T. and Mizuta, Y. (2005). Continuity of weakly monotone Sobolev functions of variable exponent. *Adv. Math. Sci. Appl., Maruzen Co. Ltd*, **15**, 571–585.
37. Futamura, T. and Mizuta, Y. (2005). Continuity properties of Riesz potentials for functions in $L^{p(\cdot)}$ of variable exponent. *Math. Inequal. Appl*, **8(1)**, 619–631.
38. Futamura, T. and Mizuta, Y. (2006). Maximal functions for Lebesgue spaces with variable exponent approaching 1. *Hiroshima Math. J.*, **36(1)**, 23–28.
39. Futamura, T., Mizuta, Y. and Shimomura, T. (2006). Sobolev embeddings for Riesz potential space of variable exponent. *Math. Nachr.*, **31(279)**, 495–522.
40. Futamura, T., Mizuta, Y. and Shimomura, T. (2006). Sobolev embeddings for variable exponent Riesz potentials on metric spaces. *Ann. Acad. Sci. Fenn. Math.*, **31(2)**, 495–522.
41. Gakhov, F. D. (1977). *Boundary value problems.* (Russian), 3rd ed. Moscow: Nauka, 640 pages. (Transl. of 2nd edition in Oxford: Pergamon Press, 1966, 561p.).
42. Genebashvili, I., Gogatishvili, A., Kokilashvili, V. and Krbec, M. (1998). *Weight theory for integral transforms on spaces of homogeneous type.* Pitman Monographs and Surveys, Pure and Applied mathematics: Longman Scientific and Technical, 422 pages.
43. Gohberg, I. and Krupnik, N. (1992). *One-Dimensional Linear Singular Integral equations, Vol. I. Introduction.* Operator theory: Advances and Applications, **53**. Basel-Boston: Birkhauser Verlag, 266 pages.
44. Gohberg, I. and Krupnik, N. (1992). *One-Dimensional Linear Singular Integral equations, Vol. II. General Theory and Applications.* Operator theory: Advances and Applications, **54**. Basel-Boston: Birkhauser Verlag, 232 pages.
45. Hajłasz, P. and Koskela, P. (2000). Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, **145(688)**, 1–101.
46. Harjulehto, P. and Hästö, P. (2004). A capacity approach to the Poincaré inequality in the Sobolev imbeddings in variable exponent Sobolev spaces. *Rev. Mat. Complut.*, **17(1)**, 129–146.
47. Harjulehto, P. and Hästö, P. (2006). Sobolev inequalities for variable exponents attaining the values 1 and n. *Preprint*, pages 1–11. available at *http : //www.helsinki.fi/ pharjule/varsob/publications.shtml.*
48. Harjulehto, P., Hästö, P. and Koskenoja, M. (2005). Hardy's inequality in a variable exponent Sobolev space. *Georgian Math. J.*, **12(3)**, 431–442.
49. Harjulehto, P., Hästö, P. and Latvala, V. (2006). Sobolev embeddings in metric measure spaces with variable dimension. *Math. Z.*, **254(3)**, 591–609.
50. Harjulehto, P., Hästö, P. and Pere, M. (2004). Variable Exponent Lebesgue Spaces on Metric Spaces: The Hardy-Littlewood Maximal Operator. *Real Anal. Exchange*, **30(1)**, 87–104.
51. Harjulehto, P., Hästö, P. and Pere, M. (2006). Variable exponent Sobolev spaces on metric measure spaces. *Funct. Approx. Comment. Math.*, **36**, 79–94.
52. Hästö, P. (2007). The maximal operator in Lebesgue spaces with variable exponent near 1. *Math. Nachr.*, **280(1-2)**, 74–82.

53. Heinonen, J. (2001). *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York.
54. Helson, H. and Szegö, D. (1960). A problem in prediction theory. *Ann. Math. Pura Appl.*, **51(3-4)**, 107–138.
55. Järvenpää, E., Järvenpää, M. and Llorente, M. (2004). Local dimensions of sliced measures and stability of packing dimensions of sections of sets. *Adv. Math.*, **183(1)**, 127–154.
56. Karapetiants, N. K. and Samko, N. G. (2004). Weighted theorems on fractional integrals in the generalized Hölder spaces $H_0^\omega(\rho)$ via the indices m_ω and M_ω. *Fract. Calc. Appl. Anal.*, **7(4)**, 437–458.
57. Karlovich, A. Yu. (2005). Algebras of singular integral operators on nakano spaces with Khvedelidze weights over Carleson curves with logarithmic whirl points. *Izvestija VUZov. Severo-Kavkazskii region. Estestvennie nauki, Special issue "Pseudodifferential equations and some problems of mathematical physics", dedicated to 70th birthday of Prof. I.B.Simonenko*, pages 135–142.
58. Karlovich, A. Yu. (2006). Algebras of singular integral operators with peicewise coeffiients on weighted nakano spaces. *"Operator Theory: Advances and Applications", Birkhäuser*, **171**, 323–347.
59. Karlovich, A. Yu. and Lerner, A. K. (2005). Commutators of singular integrals on generalized L^p spaces with variable exponent. *Publ. Mat.*, **49(1)**, 111–125.
60. Khabazi, M. (2004). Maximal operators in weighted $L^{p(x)}$ spaces. *Proc. A. Razmadze Math. Inst.*, **135**, 143–144.
61. Khabazi, M. (2005). The maximal operator in spaces of homogenous type. *Proc. A. Razmadze Math. Inst.*, **138**, 17–25.
62. Khvedelidze, B. V. (1975). The method of Cauchy type integrals for discontinuous boundary value problems of the theory of holomorphic functions of one variable. (Russian). *Itogi Nauki i Tehniki. Sovrem. Probl. Mat.*, **7**, 5–162.
63. Kokilashvili, V. (2005). On a progress in the theory of integral operators in weighted Banach function spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemian-Moravian Uplands, May 28 - June 2, 2004*, Math. Inst. Acad. Sci. Czech Republick, Praha, 152–175.
64. Kokilashvili, V., Paatashvili, V. and Samko, S. (2005). Boundary value problems for analytic functions in the class of Cauchy-type integrals with density in $L^{p(\cdot)}(\Gamma)$. *Bound. Value Probl.*, **(1)**, 43–71.
65. Kokilashvili, V., Paatashvili, V. and Samko, S. (2006). Boundedness in Lebesgue spaces with variable exponent of the Cauchy singular operators on Carleson curves. In Ya. Erusalimsky, I. Gohberg, S. Grudsky, V. Rabinovich, and N. Vasilevski, editors, *"Operator Theory: Advances and Applications", dedicated to 70th birthday of Prof. I.B.Simonenko.* Birkhäuser Verlag, Basel, **170**, 167–186.
66. Kokilashvili, V., Samko, N. and Samko, S. (2006). The maximal operator in variable spaces $L^{p(\cdot)}(\Omega,\rho)$. *Georgian Math. J.*, **13(1)**, 109–125.
67. Kokilashvili, V., Samko, N. and Samko, S. (2007). The Maximal Operator in Weighted Variable Spaces $L^{p(\cdot)}$. *J. Function spaces and Appl.*, **5(3)**, 299–317.
68. Kokilashvili, V., Samko, N. and Samko, S. (2007). Singular operators in variable spaces $L^{p(\cdot)}(\Omega,\rho)$ with oscillating weights. *Math. Nachrichten*, **280(9-10)**, 1145–1156.
69. Kokilashvili, V. and Samko, S. (2007). Boundedness of maximal operators and potential operators on Carleson curves in Lebesgue spaces with variable exponent. *Acta Mathematica Sinica* (submitted).
70. Kokilashvili, V. and Samko, S. (2003). On Sobolev Theorem for the Riesz type Potentials in Lebesgue Spaces with Variable Exponent. *Zeit. Anal. Anwend.*, **22(4)**.

71. Kokilashvili, V. and Samko, S. (2003). Singular Integral Equations in the Lebesgue Spaces with Variable Exponent. *Proc. A. Razmadze Math. Inst.*, **131**, 61–78.
72. Kokilashvili, V. and Samko, S. (2003). Singular Integrals in Weighted Lebesgue Spaces with Variable Exponent. *Georgian Math. J.*, **10(1)**, 145–156.
73. Kokilashvili, V. and Samko, S. (2004). Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Revista Matematica Iberoamericana*, **20(2)**, 495–517.
74. Kokilashvili, V. and Samko, S. (2005). Boundedness in Lebesgue spaces with variable exponent of maximal, singular and potential operators. *Izvestija VUZov. Severo-Kavkazskii region. Estestvennie nauki, Special issue "Pseudodifferential equations and some problems of mathematical physics", dedicated to 70th birthday of Prof. I.B.Simonenko*, 152–158.
75. Kovácǐk, O. and Rákosnǐk, J. (1991). On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.*, **41(116)**, 592–618.
76. Lerner, A. (2005). Some remarks on the Hardy-Littlewood maximal function on variable L^p spaces. *Math. Zeit.*, **251(3)**, 509–521.
77. Lerner, A. (2005). On modular inequalities in variable L^p spaces. *Arch. Mat.*, **85(6)**, 538–543.
78. Macías, R. and Segovia, C. (1981). A Well Behaved Quasidistance for Spaces of Homogeneous Type. *Trab. Mat.Inst.Argentina Mat.*, **32**, 1–18.
79. Maligranda, L. (1985). Indices and interpolation. *Dissertationes Math. (Rozprawy Mat.)*, **234**, 49.
80. Mizuta, Y. and Shimomura, T. (2004). Continuity of Sobolev functions of variable exponent on metric spaces. *Proc. Japan Acad.*, **80, Ser. A**, 96–99.
81. Mizuta, Y. and Shimomura, T. (2005). Sobolev's inequality for Riesz potentials with variable exponent satisfying a log-Hölder condition at infinity. *J. Math. Anal. Appl.*, **311**, 268–288.
82. Muskhelishvili, N. I. (1953). *Singular Integral Equations*. Groningen: Noordhoff.
83. Nekvinda, A. (2004). Hardy-Littlewood maximal operator on $L^{p(x)}(\mathbb{R}^n)$. *Math. Inequal. and Appl.*, **7(2)**, 255–265.
84. Nekvinda, A. (2007). A note on maximal operator on ℓ^{p_n} and $L^{p(x)}(\mathbb{R}^1)$. *J.Funct. Spaces and Appl.*, **5**, 49–88.
85. Rabinovich, V. S. and Samko, S. G. Boundedness and Fredholmness of pseudodifferential operators in variable exponent spaces. *Revista Matematica Iberoamericana*.
86. Rafeiro, H. and Samko, S. (2005). On multidimensional analogue of Marchaud formula for fractional Riesz-type derivatives in domains in R^n. *Fract. Calc. and Appl. Anal.*, **8(4)**, 393–401.
87. Rafeiro, H. and Samko, S. (2007). Characterization of the range of one-dimensional fractional integration in the space with variable exponent. *Proceedings of WOAT Conference, Sept 2006*, In "Operator Algebras, Operator Theory and Applications" edited by M.A. Bastos, I. Gohberg, A.B. Lebre (to appear).
88. Ružička, M. (2000). *Electroreological Fluids: Modeling and Mathematical Theory.* Springer, Lecture Notes in Math., **1748**, 176 pages.
89. Samko, N. (2008). Parameter depending almost monotonic functions and their applications to dimensions in metric measure spaces. *J. Funct. Spaces and Appl.*, (to appear).
90. Samko, N. (1999). Singular integral operators in weighted spaces with generalized Hölder condition. *Proc. A. Razmadze Math. Inst*, **120**, 107–134.
91. Samko, N. (2003). Criterion for the Fredholmness of singular operators with piecewise continuous coefficients in generalized Hölder spaces with weight. In *Singular integral operators, factorization and applications*, Birkhäuser, Basel, vol. **142** of *Oper. Theory Adv. Appl.*, 345–356.

92. Samko, N. (2004). On compactness of Integral Operators with a Generalized Weak Singularity in Weighted Spaces of Continuous Functions with a Given Continuity Modulus. *Proc. A. Razmadze Math. Inst*, **136**, 91–113.
93. Samko, N. (2005). On non-equilibrated almost monotonic functions of the Zygmund-Bary-Stechkin class. *Real Anal. Exch.*, **30(2)**, 727–745.
94. Samko, N., Samko, S. and Vakulov, B. (2007). Weighted sobolev theorem in Lebesgue spaces with variable exponent. *J. Math. Anal. and Applic.*, **335(1)**, 560–583.
95. Samko, S. (1998). Convolution and potential type operators in $L^{p(x)}$. *Integr. Transf. and Special Funct.*, **7(3-4)**, 261–284.
96. Samko, S. (2002). *Hypersingular Integrals and their Applications.* London-New-York: Taylor & Francis, Series "*Analytical Methods and Special Functions*", **5**, 358 + xvii pages.
97. Samko, S. (2003). Hardy inequality in the generalized Lebesgue spaces. *Frac. Calc. and Appl. Anal*, **6(4)**, 355–362.
98. Samko, S. (2003). Hardy-Littlewood-Stein-Weiss inequality in the Lebesgue spaces with variable exponent. *Frac. Calc. and Appl. Anal*, **6(4)**, 421–440.
99. Samko, S. (2005). On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators. *Integr. Transf. and Spec. Funct*, **16(5-6)**, 461–482.
100. Samko, S. G., Shargorodsky, E. and Vakulov, B. (2007). Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators, II. *J. Math, Anal. Appl.*, **325(1)**, 745–751.
101. Samko, S. G. and Vakulov, B. G. (2005). Weighted Sobolev theorem with variable exponent. *J. Math, Anal. Appl.*, **310**, 229–246.
102. Sharapudinov, I. I. (1979). The topology of the space $\mathcal{L}^{p(t)}([0, 1])$ (Russian). *Mat. Zametki*, **26(4)**, 613–632.
103. Simonenko, I. B. (1968). Some general questions in the theory of the Riemann boundary value problem (Russian). *Izv. Akad. Nauk SSSR, ser. Mat.*, **32(5)**, 1138–1146. (Transl. in Math. USSR Izvestija, **2** (1968), no 5, 1091-1099).
104. Xu, Jing-shi. (2007). The boundedness of multilinear commutators of singular integrals of Lebesgue spaces with variable exponent. *Czech. Math. J.*, **57(132)**, 13–27.