

APPROXIMATIVE METHOD FOR THE INVERSION  
OF THE RIESZ POTENTIAL OPERATOR  
IN VARIABLE LEBESGUE SPACES \*

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*Dedicated to the 80th anniversary  
of Professor Paul Butzer*

**Abstract**

We consider the problem of inversion of the Riesz potential operator on variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbf{R}^n)$ . To this end, we extend the so called *method of approximative inverses* known in the case of constant  $p$ , to the case of variable exponents  $p(x)$ . The main advantage of this approach is the possibility to cover the case when  $p(x)$  may take values equal to 1.

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*Key Words and Phrases:* Riesz fractional derivative, Riesz potential operator, identity approximation, approximative inverse operator, hypersingular integral

**1. Introduction**

It is well known that the operator inverse to the Riesz potential operator  $f = I^\alpha \varphi$  called also *Riesz fractional derivative*, on “nice” functions has the form  $\varphi = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}f$ , where  $\mathcal{F}$  is the Fourier transform. On “not so

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“nice” functions, for example for Lebesgue integrable functions, there are known two ways to effectively realize the construction  $\mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}f$  as an operator inverse to  $I^\alpha$  in the space  $L^p(\mathbf{R}^n)$ . One is to realize the inverse operator as a hypersingular integral (the direct approach), another one is to construct the inverse operator as the limit of “nice” convolutions (the *method of approximative inverse operators*), see [19, 21] for the former, and [13, 17], [19, Ch. 11] for the latter. The development of the method of approximative inverse operators in application to the inversion problem of potential type operators was initiated by Zavolzhenskii and Nogin [23] in the beginning of the 90’ies, see also the presentation of those results in [19, Ch. 11, §7].

Last decade or so, the theory of the generalized Lebesgue spaces  $L^{p(\cdot)}(\Omega)$  was intensively developed, inspired both by difficult open problems in this theory, and possible applications shown in [15], we refer e.g. to papers [2, 4, 7, 8, 10, 11, 12] and references therein, see also the surveying papers [5, 9, 20] on the topic.

In the case of Lebesgue spaces  $L^{p(\cdot)}(\mathbf{R}^n)$  with variable exponent  $p(x)$ , the method of hypersingular integrals for the same goals was used in [1] where it was proved that the hypersingular operator treated as convergent in the norm of the space  $L^{p(\cdot)}(\mathbf{R}^n)$ , provides a real left inverse operator to the Riesz potential operator, under the log-condition on the exponent  $p(x)$ .

In this paper, we extend the method of approximative inverse operators to the case of variable Lebesgue spaces.

Note that the method of approximative inverse operators is applicable to the Riesz potential operators of complex order  $\alpha$  with  $\Re\alpha > 0$ . This means that in terms of approximation we in fact construct fractional powers of complex order of the minus Laplace operator. In comparison, observe that the direct inversion by means of hypersingular integrals works for complex values of  $\alpha$  only in the case  $0 < \Re\alpha < 2$ , see [19, Ch. 3].

However, within the frameworks of the variable exponent Lebesgue spaces  $L^{p(\cdot)}(\mathbf{R}^n)$  the most important advantage of the approximation inversion, in comparison with the direct inversion by hypersingular integrals, is in a different scope. As is known, Young theorem for convolutions is not valid for variable exponent spaces. One of the main tools to deal with convolution operators in these spaces was to compare convolution operators with maximal functions. This inevitably led to the restriction that  $p(x)$  may not take values  $p(x) = 1$ . (We refer e.g. to special efforts in the paper [6] devoted to Sobolev embeddings in variable exponent spaces in the cases

where  $p(x)$  may approach 1).

In this paper, basing on a recent progress in the problem of identity approximation in variable Lebesgue spaces, see [3], we show that the approximation methods works in the general case  $p(x) \geq 1$ , which was impossible to realize via the method of hypersingular integrals.

The main results are given in Section 4. In Subsection 4.1 we prove a general result on inversion in terms of an arbitrary sequence of convolutions, admissible in a sense for the approximation of the inversion, while Subsections 4.2 and 4.3 are devoted to two important realizations of the general Theorem 4.10. Sections 2 and 3 provide some necessary preliminaries.

## 2. Preliminaries

We refer to [11, 16] for details on variable Lebesgue spaces over domains in  $\mathbf{R}^n$ , but give some necessary definitions. For a measurable function  $p : \Omega \rightarrow [1, \infty)$ , where  $\Omega \subseteq \mathbf{R}^n$  is an open set, we put

$$p_+ = p_+(\Omega) := \operatorname{ess\,sup}_{x \in \Omega} p(x) \quad \text{and} \quad p_- = p_-(\Omega) := \operatorname{ess\,inf}_{x \in \Omega} p(x).$$

The generalized Lebesgue space  $L^{p(\cdot)}(\Omega)$  with variable exponent is introduced as the set of functions for which the following modular is finite

$$\varrho_p(f) := \int_{\Omega} |f(x)|^{p(x)} dx.$$

Equipped with the norm

$$\|\varphi\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \varrho_p \left( \frac{\varphi}{\lambda} \right) \leq 1 \right\},$$

this is a Banach space when  $p_+ < \infty$ . By  $p'(x)$  we denote the conjugate exponent:  $\frac{1}{p(x)} + \frac{1}{p'(x)} \equiv 1$ .

In the sequel, we use the following standard conditions on  $p(x)$ :

$$1 \leq p_- \leq p(x) \leq p_+ < \infty, \quad x \in \Omega, \quad (1)$$

or sometimes

$$1 < p_- \leq p(x) \leq p_+ < \infty, \quad x \in \Omega, \quad (2)$$

and

$$|p(x) - p(y)| \leq \frac{C}{-\ln(|x - y|)}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \Omega. \quad (3)$$

In the case  $\Omega = \mathbf{R}^n$  we also make use of the following definition.

DEFINITION 2.1. Let  $\Omega$  be an unbounded set. By  $\mathcal{P}_\infty(\Omega)$  we denote the set of all bounded measurable functions  $p : \Omega \rightarrow [1, \infty)$  which satisfy assumption (1) and the condition that there exists  $p(\infty) = \lim_{\Omega \ni x \rightarrow \infty} p(x)$  and

$$|p(x) - p(\infty)| \leq \frac{C}{\ln(2 + |x|)}, \quad x \in \Omega. \quad (4)$$

In the case  $\Omega = \mathbf{R}_+^1$ , we will also use the following definition.

DEFINITION 2.2. By  $\mathbb{P}_{0,\infty}(\mathbf{R}_+^1)$  we denote the class of exponents  $p$  satisfying assumption (1) and the condition that there exist  $p(0) = \lim_{x \rightarrow 0} p(x)$  and  $p(\infty) = \lim_{x \rightarrow \infty} p(x)$  and

$$|p(x) - p(0)| \leq \frac{C}{|\ln|x||} \text{ for } |x| \leq \frac{1}{2} \text{ and } |p(x) - p(\infty)| \leq \frac{C}{\ln|x|} \text{ for } |x| \geq 2. \quad (5)$$

## 2.1. Approximate identities

Let  $\phi$  be an integrable function such that  $\int_{\mathbf{R}^n} \phi(x) dx = 1$ . For each  $t > 0$ , we put  $\phi_t := t^{-n} \phi(xt^{-1})$ . Following [3], we say that  $\{\phi_t\}$  is a potential-type approximate identity, if the radial majorant of  $\phi$ , defined by

$$\tilde{\phi}(x) = \sup_{|y| \geq |x|} |\phi(y)|$$

is integrable. In [3] the following proposition was proved.

PROPOSITION 2.3. Given an open set  $\Omega$ , let  $p \in \mathcal{P}_\infty(\Omega)$  and satisfy log-condition (3). If  $\{\phi_t\}$  is a potential-type approximate identity, then for all  $t > 0$ , we have:

i)  $\|\phi_t * f\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}$ ,

and  $\{\phi_t * f\}$  converges to  $f$  in  $L^{p(\cdot)}(\Omega)$  norm, i.e.

ii)  $\lim_{t \rightarrow 0} \|\phi_t * f - f\|_{p(\cdot)} = 0$ .

REMARK 2.4. Convergence of potential-type approximate identities in variable exponent Lebesgue spaces  $L^{p(\cdot)}$  was known from [4] under the assumption that the maximal operator is bounded. An extension of this fact to weighted spaces was given in [14]. Proposition 2.3 does not use the information about the maximal operator and allows to include the cases where  $p(x)$  may be equal to 1.

## 2.2. Riesz potential operator

Recall that the Riesz potential operator, also known as *fractional integral operator*, is given by

$$I^\alpha g(x) := \frac{1}{\gamma_n(\alpha)} \int_{\mathbf{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}} dy, \quad (6)$$

with the normalizing constant  $\gamma_n(\alpha) = 2^\alpha \pi^{\frac{n}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}$ . We admit complex values of  $\alpha$ . Everywhere in the sequel we assume that

$$0 < \Re \alpha < n.$$

In [1] it was proved in the case of real  $\alpha$  that the operator

$$\mathbb{D}^\alpha f := \frac{1}{d_{n,\ell}(\alpha)} \int_{\mathbf{R}^n} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}} dy, \quad (7)$$

where  $(\Delta_y^\ell f)(x)$  is a centered or non-centered finite difference of order  $\ell > 2 \lfloor \frac{\alpha}{2} \rfloor$  and  $d_{n,\ell}(\alpha)$  is some normalizing constant, is left inverse to the operator  $I^\alpha$  also in the frameworks of variable Lebesgue spaces  $L^{p(\cdot)}(\mathbf{R}^n)$ , see [19, 21] for the case of constant  $p$ .

The integral in (7) is known as *Riesz fractional derivative*. When considered on functions in the range  $I^\alpha(X)$  of the operator  $I^\alpha$  over this or that space  $X$ , it is always interpreted as the limit, in the norm of the space  $X$ , of the truncated operators  $\mathbb{D}_\varepsilon^\alpha$ , i.e.  $\mathbb{D}^\alpha f := \lim_{\substack{\varepsilon \rightarrow 0 \\ (X)}} \mathbb{D}_\varepsilon^\alpha f$ , where

$$\mathbb{D}_\varepsilon^\alpha f = \frac{1}{d_{n,\ell}(\alpha)} \int_{|y|>\varepsilon} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}} dy.$$

The following proposition was proved in [1, Theo. 5.5]

**PROPOSITION 2.5.** *Let  $1 < p_-(\mathbf{R}^n) \leq p_+(\mathbf{R}^n) < \frac{n}{\alpha}$ . If the maximal operator is bounded in the space  $L^{p(\cdot)}(\mathbf{R}^n)$ , then*

$$\mathbb{D}^\alpha I^\alpha \varphi = \varphi, \quad \varphi \in L^{p(\cdot)}(\mathbf{R}^n),$$

where the hypersingular operator  $\mathbb{D}^\alpha$  is understood as convergent in  $L^{p(\cdot)}$ -norm.

REMARK 2.6. As in the case of constant  $p$  (see [19, Ch. 3]), Proposition 2.5 remains valid for complex  $\alpha$  in the region  $0 < \Re \alpha < 2$  with the condition  $p_+(\mathbf{R}^n) < \frac{n}{\alpha}$  replaced by  $p_+(\mathbf{R}^n) < \frac{n}{\Re \alpha}$ . The method of approximative inverse operators presented in the sequel allows to consider all complex values of  $\alpha$  with  $0 < \Re \alpha < n$ .

### 3. The method of approximative inverse operators

We refer to [17], [19, Ch. 11] for more details on approximative inverse operators, but recall some principal ideas. Given a convolution operator  $A\varphi = a * \varphi$ , we have

$$\mathcal{F}(A\varphi)(\xi) = \widehat{a}(\xi) \cdot \widehat{\varphi}(\xi), \quad (8)$$

where

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{ix \cdot \xi} f(x) dx$$

is the Fourier transform of  $f$ . In the formal inversion

$$A^{-1}\varphi = \mathcal{F}^{-1} \left( \frac{1}{\widehat{a}(\xi)} \cdot \widehat{\varphi}(\xi) \right) \quad (9)$$

we encounter the problem that the factor  $[\widehat{a}(\xi)]^{-1}$  usually increases at infinity. To overcome this problem, we may try to interpret (9) as

$$(A^{-1})\varphi = \lim_{\varepsilon \rightarrow 0} (A^{-1})_\varepsilon \varphi = \lim_{\varepsilon \rightarrow 0} \mathcal{F}^{-1} \left( \frac{m_\varepsilon(\xi)}{\widehat{a}(\xi)} \cdot \widehat{\varphi}(\xi) \right), \quad (10)$$

where we may choose a “nice” factor  $m_\varepsilon(\xi)$  with compact support or rapidly decaying at infinity, and such that  $m_\varepsilon(\xi)$  tends to 1 as  $\varepsilon \rightarrow 0$ . Then one needs to justify that under this or that choice of  $m_\varepsilon(\xi)$ , the construction defined in (10) will really generate the inverse operator in the space under consideration.

In the case of the Riesz potential operator (6), in view of homogeneity of the kernel, this idea leads to the following form of the the inverse operator

$$(I^\alpha)^{-1} f = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^{n+\alpha}} \int_{\mathbf{R}^n} k_\alpha \left( \frac{y}{\varepsilon} \right) f(x-y) dy,$$

where the kernel  $k_\alpha(x) \in L^1(\mathbf{R}^n)$  is to be looked for, see details in [17], [19, Ch. 11].

### 3.1. General requirements to the kernel $k_\alpha(y)$

In Fourier transform we have

$$\mathcal{F}[(I^\alpha)^{-1}f] = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^\alpha} \widehat{k_\alpha}(\varepsilon \xi) \cdot \widehat{f}(\xi).$$

According to the idea in (10), the kernel  $k_\alpha(x)$  should satisfy the condition

$$\frac{1}{\varepsilon^\alpha} \widehat{k_\alpha}(\varepsilon \xi) \rightarrow |\xi|^\alpha \quad \text{as} \quad \varepsilon \rightarrow 0,$$

which is equivalent to

$$\widehat{k_\alpha}(\xi) = |\xi|^\alpha \mathcal{K}(\xi) \quad (11)$$

where  $\lim_{\xi \rightarrow 0} \mathcal{K}(\xi) = 1$ . If we take an arbitrary “nice” function  $\mathcal{K}(\xi)$  rapidly vanishing at infinity and  $\mathcal{K}(0) = 1$ , then  $k_\alpha(x) \in L^1(\mathbf{R}^n)$ . In Fourier pre-images we have from (11)

$$k(x) = I^\alpha k_\alpha, \quad (12)$$

where  $k(x) = (\mathcal{F}^{-1}\mathcal{K})(x)$ , and

$$\int_{\mathbf{R}^n} k(x) dx = 1. \quad (13)$$

In other words, the kernel  $k_\alpha(x)$  we are looking for, should be the Riesz fractional derivative of an identity approximation kernel  $k(x)$ .

Observe that under concrete choices of  $k_\alpha$ , the relation

$$k_\alpha(y) = \mathcal{O}\left(\frac{1}{|y|^{n+\Re\alpha}}\right) \quad \text{as} \quad |y| \rightarrow \infty \quad (14)$$

usually holds, see [19, p. 317].

The following two lemmas were proved in [17], see also their presentation in [19, p. 317].

LEMMA 3.7. *Let*

$$y^j k_\alpha(y) \in L^1(\mathbf{R}^n), \quad 0 \leq |j| < \Re\alpha. \quad (15)$$

*Then the condition*

$$\int_{\mathbf{R}^n} y^j k_\alpha(y) dy = 0, \quad 0 \leq |j| < \Re\alpha, \quad (16)$$

*is necessary and sufficient for the existence of limit (11) on nice functions  $f \in \mathcal{S}$ , say.*

LEMMA 3.8. *Let  $k(x) \in L^1(\mathbf{R}^n)$ , let  $x^j k(x) \in L^1(\mathbf{R}^n)$  for all  $j$  with  $|j| < \Re\alpha$  and let  $k(x)$  have the Riesz derivative  $\mathbb{D}^\alpha k(x) = \lim_{\substack{\varepsilon \rightarrow 0 \\ (L^1)}} \mathbb{D}_\varepsilon^\alpha k(x)$ . If besides this,  $k_\alpha(x) = \mathbb{D}^\alpha k(x)$  itself satisfies the condition (15), then equation (16) is satisfied.*

DEFINITION 3.9. The identity approximation kernel  $k(x)$  is called admissible for the inversion of the Riesz operator  $I^\alpha$ , if

$$k(x) \in L^1(\mathbf{R}^n) \cap I^\alpha(L^1). \quad (17)$$

#### 4. Inversion of the Riesz potential operator

##### 4.1. Inversion with the usage of an arbitrary admissible approximation kernel $k_\alpha(x)$

THEOREM 4.10. Let  $f(x) = I^\alpha \varphi$ , where  $\varphi \in L^{p(\cdot)}(\mathbf{R}^n)$ . Suppose that the exponent  $p(\cdot) \in \mathcal{P}_\infty(\mathbf{R}^n)$  satisfies the log-condition (3) and  $1 \leq p_- \leq p_+ < \frac{n}{\Re \alpha}$ . Then

$$\lim_{\substack{\varepsilon \rightarrow 0 \\ (L^{p(\cdot)})}} \frac{1}{\varepsilon^{n+\alpha}} \int_{\mathbf{R}^n} k_\alpha\left(\frac{y}{\varepsilon}\right) f(x-y) dy = \varphi(x), \quad (18)$$

where  $k(x)$  is any admissible identity approximation kernel.

Proof. Consider first  $\varphi \in \mathcal{S}$ . The equality

$$\frac{1}{\varepsilon^{n+\alpha}} \int_{\mathbf{R}^n} k_\alpha\left(\frac{y}{\varepsilon}\right) f(x-y) dy = \int_{\mathbf{R}^n} k(y) \varphi(x-\varepsilon y) dy \quad (19)$$

is valid, see [19, p. 318], which can be verified by passing to Fourier transforms.

We wish to extend this relation to functions in  $L^{p(\cdot)}(\mathbf{R}^n)$ . To this end, we may take  $\varepsilon = 1$ . Then (19) is

$$k_\alpha * I^\alpha \varphi = (I^\alpha k_\alpha) * \varphi, \quad (20)$$

where (12) was taken into account. Since

$$L^{p(\cdot)}(\mathbf{R}^n) \subset L^{p_-}(\mathbf{R}^n) + L^{p_+}(\mathbf{R}^n),$$

to verify that (20) is valid for  $\varphi \in L^{p(\cdot)}(\mathbf{R}^n)$ , it suffices to know that it is valid for functions  $\varphi \in L^{p_+}(\mathbf{R}^n)$  and  $\varphi \in L^{p_-}(\mathbf{R}^n)$ , which is easy to check in the case of constant exponents, we refer to the corresponding details in [17, p. 235]. Relation (19) having been proved for  $\varphi \in L^{p(\cdot)}(\mathbf{R}^n)$ , to derive the statement of the theorem from (19), it suffices to make use of Proposition 2.3.  $\blacksquare$

#### 4.2. Inversion with Poisson kernel as approximation identity

Since the Poisson kernel

$$P(x, t) = \frac{c_n t}{(|x|^2 + t^2)^{\frac{n+1}{2}}}; \quad c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}, \quad (21)$$

generates a well-known identity approximation, our first choice is  $k_\alpha(x) = \mathbb{D}^\alpha k$  with  $k(x) = P(x, 1)$ . As shown in [19, Ch.11], in this case

$$k_\alpha(x) = \frac{\Gamma(n + \alpha)}{2^{n-1} \pi^{\frac{n}{2}} \Gamma(\frac{n}{2})} F\left(\frac{n + \alpha}{2}, \frac{n + \alpha + 1}{2}; \frac{n}{2}; -|x|^2\right), \quad (22)$$

where  $F(a, b; c; z)$  is the Gauss hypergeometric function, and

$$|k_\alpha(x)| \leq \frac{C}{(1 + |x|)^{n + \Re\alpha}}. \quad (23)$$

Observe that in the case of odd dimensions  $n = 2k + 1$ , the kernel  $k_\alpha(x)$  is an elementary function, see [19, p. 320]. In [17], see also [19], it was proved that  $k(x) = P(x, 1)$  is an admissible approximation kernel (see the proof of Theorem 11.10). By (23), the kernel  $k(x)$  satisfies then the assumptions of Theorem 4.10. We arrive at the following particular case of Theorem 4.10 (obtained in [22] in the case of constant  $p$ ).

**THEOREM 4.11.** *Let  $p \in \mathcal{P}_\infty(\mathbf{R}^n)$  satisfy log-condition (3) and  $1 \leq p_- \leq p_+ < \frac{n}{\Re\alpha}$ . Then inversion (18) of the Riesz potential operator holds under the choice (22) of  $k_\alpha(x)$ .*

**REMARK 4.12.** The same is valid if one takes  $k(x) = W(x, 1)$ , making use of another famous approximation, Gauss-Weierstraß kernel  $W(x, t)$ . In this case the kernel  $k_\alpha(x)$  is expressed in terms of confluent hypergeometric function, see [19, p. 321].

#### 4.3. Inversion with a direct choice of $k_\alpha(x)$

In [17], see also [19, p. 324], it was shown that the following construction

$$k_\alpha(x) = \frac{1}{\gamma_n(-\alpha)} \left[ \frac{1}{(1 + |x|^2)^{\frac{n+\alpha}{2}}} - \sum_{k=1}^m \frac{(-1)^{k-1} c_{m,k}}{(1 + |x|^2)^{\frac{n+\alpha}{2} + k}} \right], \quad (24)$$

where  $m = \left[ \frac{\Re\alpha}{2} \right]$  and

$$c_{m,k} = \binom{m}{k} \frac{\left(\frac{n+1}{2}\right)_k}{\left(\frac{\alpha}{2} - m + 1\right)_k}, \quad (25)$$

satisfies the necessary condition that  $I^\alpha k_\alpha$  is an admissible approximation kernel. In particular, in the case  $0 < \Re\alpha < 2$  we have

$$k_\alpha(x) = \frac{1}{\gamma_n(-\alpha)} \left[ \frac{1}{(1+|x|^2)^{\frac{n+\alpha}{2}}} - \frac{n+\alpha}{\alpha(1+|x|^2)^{\frac{n+\alpha}{2}+1}} \right]. \quad (26)$$

Applying Theorem 4.10 with  $k_\alpha$  defined by (24), we arrive at the following statement.

**THEOREM 4.13.** *Let  $p(\cdot) \in \mathcal{P}_\infty(\mathbf{R}^n)$  satisfy log-condition (3) and  $1 \leq p_- \leq p_+ < \frac{n}{\Re\alpha}$ . Then for all  $\alpha \neq 2, 4, 6, \dots$ , the inversion of the Riesz potential operator  $f = I^\alpha \varphi$  with  $\varphi \in L^{p(\cdot)}(\mathbf{R}^n)$  can be written in the form*

$$\varphi(x) = \frac{1}{\gamma_n(-\alpha)} \lim_{\substack{\varepsilon \rightarrow 0 \\ (L^{p(\cdot)})}} \int_{\mathbf{R}^n} \left[ \frac{1}{(|y|^2 + \varepsilon^2)^{\frac{n+\alpha}{2}}} - \varepsilon \mathbb{A}(y, \varepsilon) \right] f(x - y) dy, \quad (27)$$

where

$$\mathbb{A}(y, \varepsilon) = \sum_{k=1}^m (-1)^{k-1} \frac{c_{m,k} \varepsilon^{k-1}}{(|y|^2 + \varepsilon^2)^{\frac{n+\alpha}{2}+k}},$$

$m = \left[ \frac{\Re\alpha}{2} \right]$  and  $c_{m,k}$  are given by (25). Formula (27) may be represented also in a compact form as

$$\varphi(x) = \frac{(-1)^m}{\gamma_n(2m - \alpha)} \lim_{\substack{\varepsilon \rightarrow 0 \\ (L^{p(\cdot)})}} \int_{\mathbf{R}^n} \Delta^m \left( \frac{1}{(|y|^2 + \varepsilon^2)^{\frac{n+\alpha}{2}-m}} \right) f(x - y) dy. \quad (28)$$

**P r o o f.** The statement of the theorem in form (27) follows immediately from Theorem 4.10. For formula (28) we have only to refer to the coincidence

$$\frac{(-1)^m}{\gamma_n(2m - \alpha)} \Delta^m \left( \frac{1}{(|y|^2 + \varepsilon^2)^{\frac{n+\alpha}{2}-m}} \right) = \frac{1}{\gamma_n(-\alpha)} \left[ \frac{1}{(|y|^2 + \varepsilon^2)^{\frac{n+\alpha}{2}}} - \varepsilon \mathbb{A}(y, \varepsilon) \right],$$

■

**REMARK 4.14.** The cases  $\alpha = 2, 4, \dots$  excluded in Theorem 4.13, correspond to the cases when the operator inverse to the Riesz potential operator, is nothing else but the power  $(-\Delta)^{\frac{\alpha}{2}}$  of minus Laplacean.

**COROLLARY 4.15.** *Let  $0 < \Re\alpha < 2$  and  $p$  satisfy the assumptions of Theorem 4.13. The inversion of the Riesz potential operator  $I^\alpha$  may be taken in the form*

$$\varphi(x) = \frac{1}{\gamma_n(-\alpha)} \lim_{\substack{\varepsilon \rightarrow 0 \\ (L^{p(\cdot)})}} \int_{\mathbf{R}^n} \left[ \frac{1}{(|y|^2 + \varepsilon^2)^{\frac{n+\alpha}{2}}} - \frac{(n+\alpha)\alpha^{-1}\varepsilon}{(|y|^2 + \varepsilon^2)^{\frac{n+\alpha}{2}+1}} \right] f(x - y) dy.$$

We use this opportunity to note that *the above formula given first in [17], contained a misprint there: the constant factor  $(n+1)$  in formula (7.21) in [17] should be read as  $(n+\alpha)$ .*

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