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# Singular Integrals in Weighted Lebesgue Spaces with Variable Exponent

by

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## Abstract

In the weighted Lebesgue space with variable exponent the boundedness of Calderón-Zygmund operator is established. The variable exponent  $p(x)$  is assumed to satisfy the logarithmic Dini condition and exponent  $\beta$  of the power weight  $\rho(x) = |x-x_0|^\beta$  is related only to the value  $p(x_0)$ . The mapping properties of Cauchy singular integrals defined on Lyapunov curve and on curves of bounded rotation are also investigated within the framework of the above-mentioned weighted space.

Key words: *variable exponent, singular integral operators, Lyapunov curve, curve of bounded rotation*

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## 1 Introduction

The generalized Lebesgue spaces  $L^{p(\cdot)}(\Omega)$  and the related Sobolev type spaces  $W^{m,p(x)}(R^n)$  variable exponent proved to be an appropriate tool to study models with non-standard local growth (in elasticity theory, fluid mechanics, differential equations, see for example Ružička [18], [6] and references therein).

These applications stimulate a quickly developing progress in the theory of the spaces  $L^{p(\cdot)}(\Omega)$  and  $W^{m,p(x)}(R^n)$ . We mention the papers Sharapudinov [23] (1979), [24] (1996), Kováčik, Rákosník [17] (1991), Edmunds, Rákosník

[10] (1992), Samko [19]-[20] (1998), [21] (1999), Edmunds, Lang, Nekvinda [9] (1999), Cruz-Uribe, Fiorenza, Neugebauer [3] (2002), Diening [4] - [5] (2002), Diening, Ružička [6] (2002), Edmunds, Nekvinda [8] (2002), Fiorenza [11] (2002), Kokilashvili, Samko [13] - [16] (2002), see also references therein.

Although the spaces  $L^{p(\cdot)}(\Omega)$  possess some undesirable properties (functions from these spaces are not  $p(x)$ -mean continuous, the space  $L^{p(\cdot)}(\Omega)$  is not translation invariant, convolution operators in general do not behave well and so on), there is an evident progress in their study, stimulated by applications, first of all for continuous exponents  $p(x)$  satisfying the logarithmic Dini condition. We mention in particular the result on denseness of  $C_0^\infty$ -functions in the Sobolev space  $W^{m,p(x)}(R^n)$ , see [21], and the breakthrough connected with the study of maximal operators in [4], [5].

Because of applications, a reconsideration of the main theorems of harmonic analysis is actual, with the aim to find out what theorems remain valid for variable exponents, or to find their substituting analogs. Among the challenging problems there were: the Sobolev type theorem for the Riesz potential operator  $I^\alpha$  and boundedness of singular integral operators. The Sobolev type theorem for bounded domains was proved in [19] conditionally, under the assumption that the maximal operator is bounded in the spaces  $L^{p(\cdot)}$ , which turns to be unconditional after the result of [4] - [5] on maximal operators (we refer also to [3] for maximal operators on unbounded domains).

Singular operators within the framework of the spaces with variable exponents were treated in [16] and [6].

The main goal of the present paper is to establish the boundedness of Calderon-Zygmund singular operators in weighted spaces  $L_\rho^{p(\cdot)}$ . In particular, we obtain a weighted mapping theorem for finite Hilbert transform and apply this result to the boundedness of Cauchy singular operators on curves in the complex plane.

## 2 Preliminaries.

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$  and  $p(x)$  a measurable function on  $\Omega$  such that

$$1 < \underline{p} \leq p(x) \leq \bar{p} < \infty, \quad x \in \Omega \quad (2.1)$$

and

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}, \quad |x - y| \leq \frac{1}{2}, \quad x, y \in \Omega. \quad (2.2)$$

We denote by  $\mathcal{P} = \mathcal{P}(\Omega)$  the set of functions  $p(x)$  satisfying conditions (2.1)-(2.2). We refer to Appendix A for examples of non-holderian functions satisfying condition (2.2). By  $L^{p(\cdot)}$  we denote the space of functions  $f(x)$  on  $\Omega$  such that

$$A_p(f) = \int_{\Omega} |f(x)|^{p(x)} dx < \infty.$$

This is a Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : A_p \left( \frac{f}{\lambda} \right) \leq 1 \right\} \quad (2.3)$$

(see e.g. [5]). We denote

$$\frac{1}{q(x)} = 1 - \frac{1}{p(x)}.$$

Under condition (2.1) the space  $L^{p(\cdot)}$  coincides with the space

$$\left\{ f(x) : \left| \int_{\Omega} f(x) \varphi(x) dx \right| < \infty \quad \text{for all} \quad \varphi(x) \in L^{q(\cdot)}(\Omega) \right\} \quad (2.4)$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}} \sim \sup_{\|\varphi\|_{L^{q(\cdot)}} \leq 1} \left| \int_{\Omega} f(x) \varphi(x) dx \right| \sim \sup_{A_q(\varphi) \leq 1} \left| \int_{\Omega} f(x) \varphi(x) dx \right|, \quad (2.5)$$

see [17], Theorem 2.3 or [20], Theorem 3.5.

Let  $\rho$  be a measurable almost everywhere positive integrable function. Such functions usually are called weights. The weighted Lebesgue space  $L_{\rho}^{p(\cdot)}$  is defined as the set of all measurable functions for which

$$\|f\|_{L_{\rho}^{p(\cdot)}} = \|\rho f\|_{L^{p(\cdot)}} < \infty.$$

The space  $L_{\rho}^{p(\cdot)}$  is a Banach space.

We deal with the following integral operators:  
Calderon-Zygmund singular operator

$$Tf(x) = \int_{\Omega} K(x, y) f(y) dy, \quad (2.6)$$

(as treated in [6]), the maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad (2.7)$$

the Riesz potential operator

$$I_\alpha f(x) = \int_{\Omega} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad \alpha > 0$$

and the Cauchy singular operator

$$S_\Gamma f(t) = \int_{\Gamma} \frac{f(\tau)d\tau}{\tau-t}, \quad t = t(s), \quad 0 \leq s \leq \ell \quad (2.8)$$

along a finite rectifiable Jordan curve  $\Gamma$  of the complex plane on which the arc-length is chosen as a parameter starting from any fixed point.

In definition of the maximal function we assume that  $f(x) = 0$  when  $x \notin \Omega$ .

In [5] the boundedness of the maximal operator in the space  $L^{p(\cdot)}$  was proved. Later in [6] the analogous result for Calderon-Zygmund operator (2.6) was obtained.

The boundedness of the maximal operator  $M$  in the weighted Lebesgue space  $L_\rho^{p(\cdot)}$  with the power weight  $\rho(x) = |x-x_0|^\beta$  was established by the authors in [15], see also [13]. The main point of the result in [13], [15] is that the exponent  $\beta$  is related to the value of  $p(x)$  at the point  $x_0$ . Recently we established also the boundedness of various integral operators in particular, Calderon-Zygmund operators, in weighted Lorentz type spaces with variable exponent [15], see also [14]. However, the result of [15], [14] does not imply the boundedness of singular operators in the Lebesgue spaces with variable exponent.

### 3 Statements of the main results.

Let

$$T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$$

be the maximal singular operator, where  $T_\varepsilon f(x)$  is the usual truncation

$$T_\varepsilon f(x) = \int_{|x-y|\geq\varepsilon} K(x,y) f(y) dy$$

and we assume that  $f(x) = 0$  outside  $\Omega$ . In what follows,

$$\rho(x) = \prod_{k=1}^m |x - a_k|^{\beta_k}, \quad (3.1)$$

where  $a_k \in \overline{\Omega}$ ,  $k = 1, \dots, m$ .

**Theorem 1.** *Let  $p(x) \in \mathcal{P}(\Omega)$  and  $\rho(x)$  be weight function (3.1). Then the operators  $T$  and  $T^*$  are bounded in the space  $L_\rho^{p(\cdot)}(\Omega)$  if*

$$-\frac{n}{p(a_k)} < \beta_k < \frac{n}{q(a_k)}, \quad k = 1, \dots, m. \quad (3.2)$$

Besides the operator (2.8), we also consider the corresponding maximal singular operator

$$S^* f(t) = \sup_{\varepsilon > 0} \left| \int_{|s-\sigma|>\varepsilon} \frac{f[\tau(\sigma)]\tau'(\sigma)}{\tau(\sigma) - \tau(s)} d\sigma \right| \quad (3.3)$$

where it is supposed that  $f[t(\sigma)] = 0$  when  $s \notin [0, \ell]$ .

We remind that  $\Gamma$  is called Lyapunov curve if  $t'(s) \in Lip \gamma$ ,  $0 < \gamma \leq 1$  and that in this case

$$\frac{t'(s)}{t(\sigma) - t(s)} = \frac{1}{\sigma - s} + h(s, \sigma), \quad \text{with} \quad |h(s, \sigma)| \leq \frac{c}{|\sigma - s|^{1-\gamma}} \quad (3.4)$$

see [12]; observe that

$$h(s, \sigma) = \frac{1}{t(\sigma) - t(s)} \int_0^1 [t'(s) - t'(s + \xi(\sigma - s))] d\xi \quad (3.5)$$

from which the bound for  $h(s, \sigma)$  follows.

If  $t'(s)$  is a function of bounded variation,  $\Gamma$  is called a curve of bounded rotation. When  $\Gamma$  is a curve of bounded rotation without cusps,  $\Gamma$  satisfies the chord-arc condition

$$\left| \frac{t(s) - t(\sigma)}{s - \sigma} \right| \geq m > 0. \quad (3.6)$$

Thus we have  $|t(s) - t(s_0)| \approx |s - s_0|$ .

When dealing with the operators  $S$  and  $S^*$ , we assume the functions  $p(s)$  and  $\rho(s) \geq 0$  to be defined on  $[0, \ell]$  and put

$$L_\rho^{p(\cdot)} = \{f : \|f[t(s)]\rho(s)\|_{L^{p(s)}} < \infty\}.$$

In the next theorem we take

$$\rho(s) = \prod_{k=1}^m |t(s) - t(c_k)|^{\beta_k} \approx \prod_{k=1}^m |s - c_k|^{\beta_k} \quad (3.7)$$

where  $c_k \in [0, \ell]$ ,  $k = 1, 2, \dots, m$ .

**Theorem 2.** *Let  $\Gamma$  be a Lyapunov curve or a curve of bounded rotation without cusps and let  $p(s) \in \mathcal{P}$ . The operators  $S_\Gamma$  and  $S^*$  are bounded in the space  $L_\rho^{p(\cdot)}(\Gamma)$  with the weight function (3.7) if and only if*

$$-\frac{1}{p(c_k)} < \beta_k < \frac{1}{q(c_k)}, \quad k = 1, 2, \dots, m. \quad (3.8)$$

## 4 Auxiliary results.

In this section we present some basic results which we need to prove our main statements. Let

$$M^\beta f(x) = \sup_{r>0} \frac{|x - x_0|^\beta}{|B(x, r)|} \int_{B(x, r)} \frac{|f(y)|}{|y - x_0|^\beta} dy, \quad (4.1)$$

where  $x_0 \in \overline{\Omega}$ .

**Theorem A** ([15]). *Let  $p(x) \in \mathcal{P}$ . The operator  $M^\beta$  with  $x_0 \in \Omega$  is bounded in the space  $L^{p(\cdot)}(\Omega)$  if and only if*

$$-\frac{n}{p(x_0)} < \beta < \frac{n}{q(x_0)}. \quad (4.2)$$

When  $x_0 \in \partial\Omega$ , condition (4.2) is sufficient in the case of any point  $x_0$  and necessary if the point  $x_0$  satisfies the condition  $|\Omega_r(x_0)| \sim r^n$ , where  $\Omega_r(x_0) = \{y \in \Omega : r < |y - x_0| < 2r\}$ .

**Theorem B** ([15]). *Let  $p(x) \in \mathcal{P}$ . The Riesz potential operator  $I_\alpha$  acts boundedly from the space  $L_\rho^{p(\cdot)}(\Omega)$  with the weight  $\rho(x) = |x - x_0|^\beta$ ,  $x_0 \in \Omega$ , into itself, if condition (4.2) is satisfied.*

Let  $F \in L_{loc}(\mathbb{R}^1)$  and

$$F^\#(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |F(y) - F_{B(x, r)}| dy, \quad (4.3)$$

where

$$F_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(z) dz.$$

**Proposition A** [1]. *Let  $T$  be a Calderon-Zygmund operator. Then for arbitrary  $s$ ,  $0 < s < 1$ , there exists a constant  $c_s > 0$  such that*

$$[(|Tf|^s)^\#(x)]^{\frac{1}{s}} \leq c_s Mf(x)$$

for all  $f \in C_0^\infty(\mathbb{R}^n)$ ,  $x \in \mathbb{R}^n$ .

The following statement holds (see [6], Lemma 3.5):

**Proposition B.** *Let  $p(x) \in \mathcal{P}$ . Then for all  $f \in L^{p(\cdot)}(\Omega)$  and  $g \in L^{q(\cdot)}(\Omega)$  there holds*

$$\left| \int_{\Omega} f(x)g(x) dx \right| \leq c \int_{\Omega} f^\#(x)Mg(x) dx$$

with a constant  $c > 0$  not depending on  $f$ .

**Lemma 4.1.** *Let  $p(x) \in \mathcal{P}$ ,  $w(x) = |x - x_0|^\gamma$ ,  $x_0 \in \overline{\Omega}$ ,  $-\frac{n}{p(x_0)} < \gamma < \frac{n}{q(x_0)}$ . Then*

$$\|fw\|_{L^{p(\cdot)}} \leq c \|f^\# w\|_{L^{p(\cdot)}}$$

with a constant  $c > 0$  not depending on  $f$ .

Proof. By (2.5) we have

$$\|fw\|_{L^{p(\cdot)}} \leq c \sup_{\|g\|_{L^{q(\cdot)}} \leq 1} \left| \int_{\Omega} f(x)g(x)w(x) dx \right|.$$

According to Proposition B,

$$\|fw\|_{L^{p(\cdot)}} \leq c \sup_{\|g\|_{L^{q(\cdot)}} \leq 1} \left| \int_{\Omega} f^{\#}(x)w(x)[w(x)]^{-1}M(gw) \, dx \right|.$$

Making use of the Hölder inequality for  $L^{p(\cdot)}$ , we derive

$$\|fw\|_{L^{p(\cdot)}} \leq c \sup_{\|g\|_{L^{q(\cdot)}} \leq 1} \|f^{\#}w\|_{L^{p(\cdot)}} \|w^{-1}M(gw)\|_{L^{q(\cdot)}}.$$

We observe that  $-\frac{1}{q(x_0)} < -\gamma < \frac{1}{p(x_0)}$ . Therefore, we may apply Theorem A for the space  $L^{q(\cdot)}$  with  $\beta = \gamma$  and conclude that

$$\|fw\|_{L^{p(\cdot)}} \leq \sup_{\|g\|_{L^{q(\cdot)}} \leq 1} \|f^{\#}w\|_{L^{p(\cdot)}} \|g\|_{L^{q(\cdot)}} \leq \|f^{\#}w\|_{L^{p(\cdot)}}.$$

□

Lemma 4.1 in the case of constant  $p(\cdot)$  and  $w \equiv 1$  is well known [25]. For variable exponent and  $w \equiv 1$  it was proved in [6].

**Theorem 4.2.** *Let  $p(x)$  be a measurable function on  $\mathbb{R}^n$  such that  $1 \leq \underline{p} \leq p(x) < \bar{p} < \infty$ ,  $\rho(x) \geq 0$  and  $|\{x \in \mathbb{R}^n : \rho(x) = 0\}| = 0$  and*

$$w(x) = [\rho(x)]^{p(x)} \in L_{loc}^1(\mathbb{R}^n). \quad (4.4)$$

*Then  $C_0^{\infty}(\mathbb{R}^n)$  is dense in the space  $L_{\rho}^{p(\cdot)}(\mathbb{R}^n)$ .*

**Proof. I.** First we prove that the class  $C_0(\mathbb{R}^n)$  of continuous functions with compact support is dense in the space  $L_{\rho}^{p(\cdot)}(\mathbb{R}^n)$ .

Let  $f \in L_{\rho}^{p(\cdot)}(\mathbb{R}^n)$ . Since  $|\{x \in \mathbb{R}^n : \rho(x) = 0\}| = 0$ , the function  $f(x)$  is a.e. finite.

**1st step.** The functions  $f_N(x) = \begin{cases} f(x), & |x| < N \\ 0, & |x| > N \end{cases}$  approximate  $f$  in  $L_{\rho}^{p(\cdot)}$ , since  $A_p(\rho|f - f_N|) \rightarrow 0$  as  $N \rightarrow \infty$ . Therefore, there exists a function  $g \in L_{\rho}^{p(\cdot)}$  with compact support such that  $\|f - g\|_{L_{\rho}^{p(\cdot)}} < \varepsilon$ .

**2nd step.** The function  $g$  may be approximated in  $L_{\rho}^{p(\cdot)}$  by the bounded functions with compact support  $\tilde{g}_N(x) = \begin{cases} g(x), & |g(x)| < N \\ 0, & |g(x)| > N \end{cases}$ . Indeed, the passage to the limit

$$A_p(\rho|g - \tilde{g}_N|) = \int_{\mathbb{R}^n} w(x)|g(x) - \tilde{g}_N(x)|^{p(x)} \, dx \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (4.5)$$

is justified by the Lebesgue dominated convergence theorem, since  $w(x)|g(x) - g_N(x)|^{p(x)} \leq 2w(x)|g(x)|^{p(x)}$  and the integrand tends to zero a.e. (at any point  $x$  at which  $f(x)$  is finite). So we choose  $\tilde{g}_N$  such that  $\|g - \tilde{g}_N\|_{L_\rho^{p(\cdot)}} < \varepsilon$ .

**3rd step.** To approximate the function  $\tilde{g}_N$  by continuous bounded functions, we choose  $\delta > 0$  so that

$$\int_E w(x) dx < \varepsilon_1 = \frac{\varepsilon}{(2N)^{\bar{p}}} \quad (4.6)$$

for any measurable set  $E \subset \text{supp } g$  with  $|E| < \delta$ , which is possible by (4.4). Then we choose a function  $\varphi(x) \in C(\mathbb{R}^n)$  with  $\|\varphi\|_C \leq N$  which coincides with  $\tilde{g}_N(x)$  everywhere except for possibly the set  $A \subset \text{supp } g$  with  $|A| < \delta$ , by Luzin theorem. Then

$$A_p(\rho|\tilde{g}_N - \varphi|) = \int_A w(x)|\tilde{g}_N(x) - \varphi(x)|^{p(x)} dx \leq (2N)^{\bar{p}} \int_A w(x) dx < \varepsilon.$$

**4th step.** It remains to approximate in  $L_\rho^{p(\cdot)}$  the function  $\varphi$  by a continuous function with compact support, which is done in the standard way by means of smooth truncation.

**II.** Approximation in  $L_\rho^{p(\cdot)}$  of a continuous function with compact support by  $C_0^\infty$ -functions may be already realized via identity approximation

$$K_t \varphi = \frac{1}{t^n} \int_{\mathbb{R}^n} a\left(\frac{y}{t}\right) \varphi(x - y) dy, \quad t > 0$$

with  $a(x) \in C_0^\infty$  and  $\int_{\mathbb{R}^n} a(x) dx = 1$ . Obviously,  $K_t \varphi \in C_0^\infty$  for  $\varphi \in C_0(\mathbb{R}^n)$  and  $|K_t \varphi - \varphi| < \varepsilon$  as  $t \rightarrow 0$  uniformly on any given compact set. Therefore,

$$A_p(\rho|K_t \varphi - \varphi|) = \int_B w(x)|K_t \varphi - \varphi(x)|^{p(x)} dx \leq \varepsilon^p \int_B w(x) dx$$

for  $t$  small enough,  $B$  being a sufficiently large ball.  $\square$

**Corollary.** *Let  $p(x)$  and  $\rho(x)$  satisfy the assumptions of Theorem 4.2 in  $\Omega$ . Then the set  $C^\infty(\overline{\Omega})$  is dense in the space  $L_\rho^{p(\cdot)}(\Omega)$ .*

Indeed, it suffices to observe that functions from  $C^\infty(\overline{\Omega})$  may be continued outside  $\Omega$  as  $C_0^\infty(\mathbb{R}^n)$ -functions, as well as  $p(x)$  and  $\rho(x)$  can be continued with preservation of their properties.

The statement of Theorem 4.2 is well known in the case of constant exponent, in the case of variable exponent and  $\rho(x) \equiv 1$  it was proved in [17].

Denseness of  $C_0^\infty(\mathbb{R}^n)$  in non-weighted Sobolev spaces  $W^{m,p(\cdot)}$  was proved in [21]-[22].

## 5 Proofs of the main results.

It suffices to deal with the weight of the form  $\rho(x) = |x - x_0|^\beta$ , because the general case (3.1) is easily reduced to this special case by separation of the points  $a_k$  by means of the partition of unity which provides the representation

$$\frac{\prod_{k=1}^m |x - a_k|^{\beta_k}}{\prod_{k=1}^m |y - a_k|^{\beta_k}} = \sum_{k=1}^m c_k(x, y) \frac{|x - a_k|^{\beta_k}}{|y - a_k|^{\beta_k}}$$

with bounded "coefficients"  $c_k(x, y)$ .

**Proof of Theorem 1.** Let us consider the operator  $T$ . Let  $f \in C^\infty(\overline{\Omega})$  and  $0 < s < 1$ . Obviously,

$$\|\rho T f\|_{L^{p(\cdot)}} = \|\rho^s |T f|^s\|_{L^{\frac{p(\cdot)}{s}}}^{\frac{1}{s}}. \quad (5.1)$$

Applying Lemma 4.1 with  $w(x) = [\rho(x)]^s$  and  $p(\cdot)$  replaced by  $\frac{p(\cdot)}{s}$ , we obtain

$$\|\rho T f\|_{L^{p(\cdot)}} \leq c \|\rho^s (|T f|^s)^\# \|_{L^{\frac{p(\cdot)}{s}}}^{\frac{1}{s}},$$

which is possible since the condition

$$-\frac{n}{\frac{p(x_0)}{s}} < s\beta < \frac{n}{\left(\frac{p(x_0)}{s}\right)'}$$

is satisfied. Thus we have

$$\|\rho T f\|_{L^{p(\cdot)}} \leq c \left\| \rho \left[ (|T f|^s)^\# \right]^{\frac{1}{s}} \right\|_{L^{p(\cdot)}} \quad (5.2)$$

because  $\|f\|_{L^{\frac{p(\cdot)}{s}}}^{\frac{1}{s}} = \||f|^{\frac{1}{s}}\|_{L^{p(\cdot)}}^{\frac{1}{s}}$ . Since a function  $f \in C^\infty(\overline{\Omega})$  may be continued outside  $\Omega$  as a  $C_0^\infty(\mathbb{R}^n)$ -function, Proposition A is applicable. Therefore, by Proposition A from estimate (5.2) we get

$$\|\rho T f\|_{L^{p(\cdot)}} \leq c \|\rho(M f)\|_{L^{p(\cdot)}}.$$

Now we apply Theorem A and conclude that

$$\|\rho T f\|_{L^{p(\cdot)}} \leq c \|\rho f\|_{L^{p(\cdot)}}$$

for all  $f \in C_0^\infty(\mathbb{R}^n)$ . Since  $C_0^\infty(\mathbb{R}^n)$  is dense in  $L_\rho^{p(\cdot)}$  by Theorem 4.2, we complete the proof of Theorem 1.

The boundedness of the operator  $T^*$  follows from the known estimate

$$T^* f(x) \leq c[M(T f)(x) + M f(x)],$$

from Theorem A and Theorem 1.

**Corollary 1.** *Let  $\Omega = [a, b]$ ,  $\rho(x) = \prod_{k=1}^m |x - a_k|^{\beta_k}$ ,  $a_k \in [a, b]$ ,  $k = 1, \dots, m$ , and  $p(x) \in \mathcal{P}$ . Then the finite Hilbert transform and its maximal version*

$$H_{[a,b]} f = \int_a^b \frac{f(y) dy}{y - x} \quad \text{and} \quad H_{[a,b]}^* = \sup_{\varepsilon > 0} \left| \int_{|y-x|>\varepsilon} \frac{f(y) dy}{y - x} \right| \quad (5.3)$$

are bounded in the space  $L_\rho^{p(\cdot)}(a, b)$ , if  $-\frac{1}{p(a_k)} < \beta_k < \frac{1}{q(a_k)}$ ,  $k = 1, 2, \dots, m$ .

**Proof of Theorem 2.** We assume the function  $p(s)$  to be defined on  $[0, l]$ . The function  $f(t(\sigma))$  will be denoted by  $f_0(\sigma)$ . In the case of Lyapunov curve we use equality (3.4) and apply Corollary 1 and Theorem B, which immediately gives the statement of Theorem 2.

Let  $\Gamma$  be a curve of bounded rotation without cusps and let  $V$  be the total variation of  $t'(s)$  on  $[0, l]$ . In this case the function  $h(s, \sigma)$  may be estimated as

$$|h(s, \sigma)| \leq c \frac{V(s) - V(\sigma)}{s - \sigma} \quad (5.4)$$

according to (3.5) and (3.6) (see [12], Chapter II, Subsection 2.3). Then we may proceed as in the proof of Theorem 3.2 in [16]:

$$\begin{aligned} \left| \int_{|s-\sigma|>\varepsilon} \frac{f_0(\sigma) d\sigma}{t(\sigma) - t(s)} \right| &\leq c \left| \int_{|s-\sigma|>\varepsilon} \frac{f_0(\sigma) d\sigma}{\sigma - s} \right| + c \int_{|s-\sigma|>\varepsilon} \frac{|f_0(\sigma)| (V(\sigma) - V(s))}{\sigma - s} d\sigma \\ &\leq c \left| \int_{|s-\sigma|>\varepsilon} \frac{f_0(\sigma) d\sigma}{\sigma - s} \right| + c V(s) \left| \int_{|s-\sigma|>\varepsilon} \frac{|f_0(\sigma)| d\sigma}{\sigma - s} \right| + c \left| \int_{|s-\sigma|>\varepsilon} \frac{|f_0(\sigma)| V(\sigma)}{\sigma - s} d\sigma \right|. \end{aligned}$$

From here by Corollary 1 and boundedness of the function  $V(s)$  we conclude that the operator  $S_\Gamma^*$  is bounded in  $L_\rho^{p(\cdot)}$ .

Let us prove the necessity part. From the boundedness of  $S_\Gamma$  in  $L_\rho^{p(s)}$  it follows that  $S_\Gamma f(t)$  exists almost everywhere for arbitrary  $f \in L_\rho^{p(s)}$ . Thus  $\rho$  should be such that  $f \in L^1(\Gamma)$  for arbitrary  $f \in L_\rho^{p(s)}$ . The function  $f = f\rho\rho^{-1}$  belongs to  $L^1(\Gamma)$  for arbitrary  $f \in L_\rho^{p(s)}$  if and only if  $\rho^{-1} \in L^{q(s)}$ , which follows from the equivalence (2.5). Then function  $\rho^{-1}(s) = |s - s_0|^{-\beta}$ ,  $s_0 \in [0, l]$ , belongs to  $L^{q(s)}[0, l]$  if and only if  $\beta < \frac{1}{q(s_0)}$ . Indeed, we have

$$|s - s_0|^{-\beta q(s)} = m(s)|s - s_0|^{-\beta q(s_0)},$$

where the function  $m(s) = |s - s_0|^{-\beta|q(s)-q(s_0)|}$  satisfies the condition

$$0 < c \leq m(s) \leq C < \infty$$

in view of (1.2). On the other hand, from  $|s - s_0|^{-\beta q(s_0)} \in L^{q(s)}$  we have  $\beta < \frac{1}{q(s_0)}$ .

The necessity of the condition  $-\frac{1}{p(s_0)} < \beta$  follows from the duality argument.

## 6 Appendix.

The following is an example of a function which satisfies condition (2.2) but is not a Hölder function:

$$p(x) = a(x) + \frac{b(x)}{\left(\ln \frac{A}{|x|}\right)^\gamma}, \quad x \in \Omega, \quad (6.1)$$

where  $a(x)$  and  $b(x)$  are Hölder functions,  $a(x) \geq 1$ ,  $b(x) \geq 0$ ,  $A > \sup_{x \in \Omega} |x|$  and  $\gamma \geq 1$ . One may write a little bit more complicated example:

$$p(x) = a(x) + \frac{b(x)}{\left(\ln \frac{A}{|x|}\right)^\gamma} \left( \ln \ln \ln \cdots \ln \frac{C}{|x|} \right)^\mu \quad (6.2)$$

with an arbitrary sufficiently large  $C > 1$  and  $\mu > 0$ , and the same assumptions on  $a(x), b(x)$ , and  $A$ , but  $\gamma > 1$ . It is also possible to take different powers of different logarithms as factors or superpositions in (6.2).

To prove condition (2.2) for functions (6.1) or (6.2) or of similar type, we are not obliged to check condition (2.2) directly. For this purpose we may use properties of continuity moduli. It suffices to deal with the case where  $a(x) \equiv 0$  and  $b(x) = 1$ , since we consider differences  $p(x) - p(y)$ .

We remind that a non-negative function  $f(t)$  on  $[0, \ell]$  is called a continuity modulus if

$$\omega(f, h) \sim f(h),$$

where  $\omega(f, h) = \sup_{\substack{|t_1 - t_2| \leq h \\ t_1, t_2 \in [0, \ell]}} |f(t_1) - f(t_2)|$ . There are known sufficient

conditions for a function  $f(x)$  to be continuity modulus, see, for example [2] or [7]:

- 1)  $f(x)$  is continuous on  $[0, \ell]$ ,
- 2)  $f(0) = 0$  and  $f(x) > 0$  for  $x > 0$ ,
- 3)  $f(x)$  is non-decreasing and  $\frac{f(x)}{x}$  is non-increasing on a neighborhood of the point  $x = 0$ . It is easy to check that functions (6.1)-(6.2) satisfy the above conditions 1)-4).

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